Topological Essence of the Concept "Limit of a Numerical Sequence" in the General Mathematics

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ARTICLE INFO	ABSTRACT
	High school mathematics includes a number of extremely important concepts, including the concept of the limit of the number sequence. However, most students when learning it do not understand it with certainty, but only accept it to apply it to solve exercises. This paper aims to give a topological essence of the concept of limit of a sequence and present some methods for teaching this concept in general school in Viet Nam. The Paper consists of four parts. Part 1 presents an introduction to the definition of "Limit of a sequence", part 2 deals with Some properties derived from definitions and notes in teaching, part 3 covers the Mathematical essence of the concept limit of a number sequence and part 4 talks about Defining topology on the set of real numbers. In each section, we include comments to help teach these notions and concepts. In each section, we make comments and observations to teach these concepts better.

1. INTRODUCTION

The concept of limit of a numerical sequence (or sequence) is taught in the math of grade 11 in Vietnam (the general educational program of Vietnam lasted 12 years), and it is proposed as follows:

1.1. Definition: A real number $S \in \mathbb{R}$ is a *limit* of a sequence $\{S_n\}$, $n \in \mathbb{N}$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left| s_n - s \right| < \varepsilon$ for all $n \geq N$.

In other words, by mathematical notation, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ $\forall_n \geq N$, $\left| s_n - s \right| < \varepsilon$.

It can be said that most students do not understand that why the definition of such a limited range, so the student all my life, they accept it without knowing why. So how to help students understand this important mathematical concept.

Mathematics is a way humans think of to reflect existence. So the concept of limits reflects something in life?

Let's start from the following paradox:

A person goes from A to B with the length d. To go to B, he/she must go through the midway point of A and B. Therefore, the distance from him/her to B is $\dfrac{d}{2}$. Next, he/she need to go through the middle of the remaining distance, and therefore, the distance from him/her to B is

$\left(\dfrac{d}{2} \right) : 2 = \dfrac{d}{2^2} = \dfrac{d}{4}$. The process continues so; we see that the distance from him/her to B is $\dfrac{d}{2^n}$. When n is great enough, $\dfrac{d}{2^n}$ will be very little, but it is always not 0. Therefore, it follows that the person can not go to B.

All the arguments are not wrong, but the results are not acceptable in our lives. So what does the conflict come from? That is because in this case, we have only four arithmetic operations (the addition, the subtraction, the multiplication, and the division) to reflect a phenomenon that life is moving from A to B. From that, we conclude, the four essential arithmetic addition, subtraction, multiplication, and division are not sufficient to reflect the phenomena of life. Therefore, we should use other mathematical concepts to reflect this existence. The idea allows the conception "number in the form $\dfrac{d}{2^n}$ with some n is great enough, it is considered as zero" (to reflect that the person went to A). But this is a saying, while math need to exact the concepts and the calculations.

Here we will present the exact mathematical concepts about things. It also helps students understand how intuitive notion of "limits of the sequence".

Recalling that $\mathbb{R}$ is denoted the set of real numbers; An open interval (or an interval) with the ends a, b are the set of numbers

$$(a, b) := \{x \in R; a < x < b\}$$

Let be S a sequence, that is:

$$S = \left\{ \frac{d}{2}; \frac{d}{2^2}; \frac{d}{2^3}; \ldots; \frac{d}{2^n}; \ldots \ldots \right\}.$$

I see that all interval (a, b) containing 0 (zero) has an intersection with S, that means $(a, b) \cap S \neq \varnothing$.

So we can say the number $\dfrac{d}{2^n}$ as zero with n large enough

if all interval (a, b) containing 0 (zero) intersects with the sequence S.

In other words, the sequence S is called the limit 0 (or towards 0) if $(a, b) \cap S \neq \varnothing$ for all interval (a, b) containing 0.

We believe that this definition of limit is easier to understand by intuitive characterítics, than the definition above.

Similarly, we say that the U= {u_n}, $n \in \mathbb{N}$ has the limit u if all interval (a, b) containing u then it has an intersection with U.

Mathematics has its language, which is the majority language of symbols, so now we will describe the contents following accounting language.

2.2.Definition: We say that the sequence S= {s_n}, $n \in \mathbb{N}$ has the limit s, denoted by $\lim_{n\to\infty} s_n = s$ if $\forall (a, b) \mathring{a}$ s then $(a, b) \cap S \neq \varnothing$.

2. SOME PROPERTIES DERIVED FROM DEFINITION 2.2 AND NOTES IN TEACHING

2.1. Proposition: A number s $\in \mathbb{R}$ is the limit of the sequence S= {s_n}, $n \in \mathbb{N}$ if the following condition holds: For each real number $\varepsilon > 0$, there exists a natural number m such that, for every natural number $n \geq m$, we have $|s_n - s| < \varepsilon$.

Proof: The statement: [each real number $\varepsilon > 0$, there exists a natural number m such that, for every natural number n $\geq$ m, we have $|s_n - s| < \varepsilon$] $\Leftrightarrow$ (by symbols)

$\Leftrightarrow \left[\forall \varepsilon > 0, \exists m \in \mathbb{N} \text{ for } n \geq m \Rightarrow |s_n - s| < \varepsilon \right]$

$\Leftrightarrow$ The statement $\left[\forall (a, b) \ni s \Rightarrow (a, b) \cap S \neq \varnothing \right]$

$\Leftrightarrow$ S has the limit s

The result following is evident

2.2.Corollary: A real number 0 is said to be a limit of a sequence $\{s_n\}_{n \in N}$ if and only if for every $\varepsilon > 0$, there exists N $\in \mathbb{N}$ such that $|s_n| < \varepsilon$ for all $n \geq N$

2.3. Corollary: The sequence $S = \{s_n\}$ *has a finite limit of s if and only if a* sequence $V = (v_n := s_n - s)$ *has limit of 0.* So to consider the limits of a series of numbers, we shall review the limit on a limited series in the 0.

Proof:

Put Q = {q_n ;n $\in \mathbb{N}$ }

with $q_n = s_n - s$ Then $\lim_{n\to\infty} s_n = s \Leftrightarrow \lim_{n\to\infty} q_n = 0$

3. MATHEMATICAL ESSENCE OF THE CONCEPT LIMIT OF A SEQUENCE

The concept limit of a function brings a deep mathematical meaning; it is formed on a different fundamental concept of mathematics, the topology concept, appeared as follows.

On the set of real numbers $\mathbb{R}$, we consider a family T of subsets A in $\mathbb{R}$ such that each A is a union of arbitrary numbers of interval in the form (a, b) above. Thus,

$$\mathcal{T} = \left\{ A \subseteq \mathbb{R}; A = \bigcup_{i \in I}(a_i, b_i) \right\}; (a_i, b_i) \text{ are intervals}$$

3.1. Theorem: The limit $\lim_{n\to\infty} s_n = 0$ if and only if all elements of the family T such that these elements contain 0 then have an intersection with S.

Proof: Because $A \in T \Rightarrow A = \bigcup_{i \in I}(a_i, b_i)$; A is a union of intervals (a_i, b_i), therefore the statement: [all elements of the family T such that these elements contain 0 them they have an intersection with S]

$\Leftrightarrow$ the message: [all intervals such that these intervals include 0 then they intersect with S] $\Leftrightarrow \lim_{n\to\infty} s_n = 0$.

3.2.Corollary: The sequence S has limit s if and only if all elements of the family T contain the number s, then these elements have an intersection with S.

3.3. Comment: The family T acts as a ruler to measure the closeness of a number s with the sequence S set of values. More precisely, we say that the sequence S gradually goes to the number s if all elements of the family T containing the number s contain are cut set S.

3.4. Theorem: Denote by $\mathcal{F} = \left\{ A \subseteq \mathbb{R} : A = \bigcup_{i \in I}(a_i, b_i) \right\}$

The family F has three foundation properties as follows:

1/ The empty set $\varnothing \in F$ and the set $\mathbb{R} \in F$;

2/ The family F is closed with the arbitrary union, i.e., with all subsets $A_j \in F$, then $\bigcup_{i \in J} A_j \in F$.

3/ The family F is closed with a finite intersection, which means that with a finite number of subsets $A_1, A_2, ..., A_k \in F$, then $\bigcup_{t=1}^{k} A_t \in F$

Proof :

1/ Because the interval

$$(a, a) = \varnothing \text{ and } \mathbb{R} = \bigcup_{n=0}^{\infty}(-n, n);$$ n are natural

numbers. Therefore it is clear that $\varnothing$ and $\mathbb{R}$ are in F .

2/ Now let $\{A_j\}_{j \in I}$ be some collection of elements in F which we can assume to be nonempty. If $p \in \cup_{j \in I} A_j$ we can choose some $j \in I$, and there exists an open ball $B \subseteq A_j$ containing p. But $B \subseteq \cup_{j \in I} A_j$ so this shows that $\cup_{j \in I} A_j \in F$.

3/ The first we see that

$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$$

Generally

$$\left(\bigcup_{i \in I}(a_i, b_i)\right) \cap \left(\bigcup_{j \in J}(c_j, d_j)\right) = \bigcup_{i \in I, j \in J}\left[(a_i, b_i) \cap (c_j, d_j)\right]$$

.

But the intersection of two intervals is an interval.

Hence, if $A_1, A_2, ..., A_k \in F$ then the intersection $A_1 \cap A_2 \cap ... \cap A_k \in F$.

3.5. Defining topology on the set of real numbers $\mathbb{R}$: A family ζ of subsets B of $\mathbb{R}$ is called a topology on $\mathbb{R}$ if the family ζ satisfies three following conditions:

1/ The empty set $\phi \in \zeta$ and the $\mathbb{R} \in \zeta$;

2/ The family ζ is closed with an arbitrary union, i.e., with all subsets $B_j \in \zeta$, then $\bigcup_{i \in J} B_j \in \zeta$.

3/ The family ζ is closed with a finite intersection, which means that the finite number of subsets $A_1, A_2, ..., A_k \in \zeta$, then $\bigcup_{t=1}^{k} B_t \in \zeta$.

3.6. Corollary: The family ζ above is a topology on $\mathbb{R}$.

3.7. Example: Family $\zeta = \{\varnothing; \mathbb{R}\}$ (only two elements are ϕ and $\mathbb{R}$) is a topology on $\mathbb{R}$, called crude topology.

3.8. Example: Family $\zeta = \{$all subsets of $\mathbb{R}\}$ is a topology on $\mathbb{R}$, it called discrete topology.

4. DEFINING THE LIMIT OF A SEQUENCE IN A TOPOLOGICAL SPACE

4.1.Definition: Suppose that on the set of real numbers $\mathbb{R}$, given a certain topology ζ. We say the sequence $S = \{s_n\}$ has a limit **s** or moves gradually towards s if every element of ζ which contains s, then this element has an intersection with S.

So the topological essence of the concept limit of a sequence is the limit in a certain topology on the set of real numbers $\mathbb{R}$.

4.2. Comment: The limit of sequence S is defined as above is the limit of the sequence S with F topology.

4.3. Proposition: If on $\mathbb{R}$, given F topology and sequence S has a limit, this limit is unique.

Proof: Supose that the sequence $\{s_n\}$, $n \in \mathbb{N}$ has two limits s_1 and s_2:

$$\lim s_n = s_1 \text{ and } \lim s_n = s_2 \text{ with } s_1 \neq s_2$$

for every ε > 0, there exist natural numbers m_1, m_2 such that when

$$n \geq m_1 \Rightarrow |s_n - s_1| < \varepsilon$$

and

$$n \geq m_2 \Rightarrow |s_n - s_2| < \varepsilon$$

We have

$$|s_1 - s_2| = |s_n - s_1 - (s_n - s_2)| \leq |s_n - s_1| + |s_n - s_2| < 2\varepsilon$$

.

When $\varepsilon \to 0$, then $s_1 - s_2 = 0$.

5. ADDING A TOPOLOGY ON THE SET OF REAL NUMBERS AND LIMIT OF A SEQUENCE

5.1. Definition: Let any positive number ε. We say that the interval (a, b) is ε- interval if its length is 2ε. We will build a topology on $\mathbb{R}$ as follows:

Put $\mathcal{M}:= \{C \subset \mathbb{R};$ with all $c \in C$, there exists ε- in the interval (a, b) such that $c \in (a, b) \subset C\}$.

5.2. Proposition: Family M is a topology on $\mathbb{R}$ coinciding with the topology F, i.e., $M = F$ (the two sets M and F are equal).

Proof: It is clear that $C \in M$ if only if $c \in C$ then C contains an interval containing c, that means if and only if C is a union of some intervals, therefore if and only if $C \in F$

5.3. Comment: The limit of sequence stated in the mathematical textbook of class 11 in Vietnam is the limit of a series for M topology and thus also the limit for F topology.

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