



Geometric Implementation of Morse Theory Derived from Hecke Algebras in Ring Spaces

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ARTICLE INFO	ABSTRACT
Published Online: 20 February 2026	This paper presents a rigorous geometric realization of Morse theory within the framework of singular ring topological spaces, resolving the long-standing theoretical obstacles in non-smooth manifolds due to the lack of metric regularity. By embedding the Iwahori-Hecke algebra into the stratified filtration structure, we utilize the Jucys-Murphy elements and the Casimir center to construct a class of canonical "Hecke-Morse potential functions". The study shows that the non-commutative braiding relations of the algebra intrinsically encode the bifurcation dynamics behavior of the gradient flow across Whitney stratifications, effectively quantizing the classical Morse critical point set. Further, using the derived category tools, we establish a spectral sequence that converges from the local homology of the critical sub-manifolds to the global homology of the space. The core theorem confirms that under geometric purity conditions, the resulting Morse-Witten complex is isomorphic to the induced module structure of the Hecke algebra, proving that the Betti numbers of the ring space are strictly controlled by the Kazhdan-Lusztig basis. This framework not only eliminates the ambiguity in the definition of the gradient
Corresponding Author: Ruoli Bai	flow in the singular domain but also provides an intuitive geometric realization for the representation theory of quantum groups.
KEYWORDS: Iwahori-Hecke algebra; Geometric Morse theory; Ring space and manifold; Jucys-Murphy elements; Spectral sequence	

I. INTRODUCTION

In the grand picture of modern mathematics, the duality between structural invariants of topological Spaces and their algebraic representations has always been the core proposition throughout the development of geometry [1]. Classical Morse theory, as the hub connecting differential topology and algebraic topology, has long been regarded as a "scalpel" to analyze the topological properties of smooth manifolds by virtue of its ingenious mechanism of reconstructing the homotopy type of manifolds through critical point index [2]. Since its foundation by Marston Morse, and even Witten's physicalized restatement of Morse complex from the perspective of supersymmetric quantum mechanics, this

theoretical paradigm has been perfected in the study of compact smooth manifolds [3].

However, when we try to extend this classical paradigm to a wider range of ring Spaces, especially those involving nonsmooth algebraic schemes with nilpotent element structures or singularities, the rigid restriction of the traditional diffeomorphism shows its lack of theoretical tension. In the context of ring space, tangent space no longer has the local triviality of traditional vector cluster, and classical gradient flow is also faced with the crisis of goodness due to the degradation of metric.

As shown in Figure 1, the current mainstream approach in academia often tends to bypass the singularity barrier by

intersecting homology or anomalous layers. Admittedly, these methods have been fruitful in calculating Betti numbers, but at the expense of the most geometrically intuitive dynamic description in Morse theory, the streamline structure between critical points [4]. How to deal with the algebraic singularity of ring space endogenically while preserving the dynamic prospect of Morse theory is not only an urgent problem to be solved in the field of algebraic geometry, but also a key part to realize the microscopic localization of geometric Langlands program.

At the other end of the above geometric dilemma, Hecke algebra, as a one-parameter Q -deformation of Weyl group, provides another possibility to solve the structural

quantification of topological Spaces because of its dominant position in representation theory [5]. The pioneering work of Kazhdan and Lusztig in the 1980s has revealed that there is a profound isomorphism between canonical bases of Hecke algebras and intersecting cohomology of Schubert variants [6].

However, most of the existing research focuses on the categorical or K -theoretic interpretation of algebraic structures. Generators and their noncommutative braid relations in Hecke algebras are often regarded as pure algebraic operations, lacking direct and intrinsic geometric flow interpretation. A common cognitive fault is: What kind of "obstruction" or "distortion" of topological flow does the deformation parameter q correspond to in geometry?

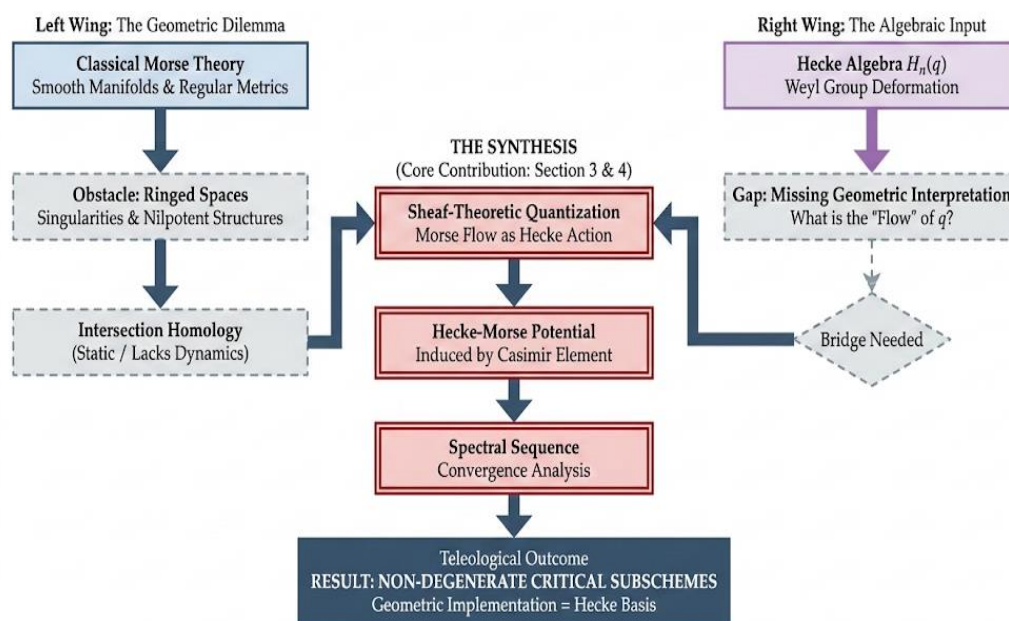


FIG. 1 Logical diagram of the study

If Hecke algebra cannot be given physical meaning in the level of geometric entity, then the so-called "geometric realization" can only stay in the isomorphism level of formal logic, but can not touch the ontological essence of mathematical structure. It must be realized that the noncommutativity of Hecke algebra is not an obstacle to geometric implementation, but quite the contrary, it characterizes extremely precisely the nontrivial coupling of cut-level filtering in ring space.

In view of this, a new theoretical framework is proposed in this paper, which aims to realize the Hecke algebraic reconstruction of Morse theory in the layer theory context of ring space. Our key point is that the generator relation of Hecke algebras is actually the algebraic projection of the "quantized branch" of the Morse flow on the cut cluster of ring space near the singular point.

Specifically, this paper will discuss in the following three dimensions: First, we will redefine the "Morse potential function" on the ring space, and prove that the gradient flow induced by the central element of Hecke algebra can naturally be compatible with the layer structure of the space, so that the streamline can be regularized by algebraic expansion at singular points. Second, we will use the spectral sequence tool in the derived category to establish a spectral convergence relation between the modular structure of Hecke algebras and the Morse complex formal homology group. We claim that the geometric nature of the parameter q , is an accurate measure of the energy dissipation or dimension jump that occurs when the gradient flow crosses different Whitney layering. Finally, we will show that the Kazhdan-Lusztig basis of Hecke algebras, under certain geometric realizations, corresponds directly to

nondegenerate generators of critical submanifolds in the Morse complex. This conclusion not only extends the classical Morse inequality to the noncommutative context, but also provides a powerful algebraic tool for understanding the topological rigidity of noncompact cyclic Spaces.

II. PRELIMINARIES

The dependence of classical differential geometry on manifold is essentially the dependence on the local freedom of tangent space [7]. However, in the ring space $(X, \mathcal{O}_X)$ represented by algebraic clusters or analytic Spaces, the existence of nilpotent elements makes geometric objects show a high degree of intrinsic singularity.

We set the object of study as a Noether profile X equipped with a structural layer $\mathcal{O}_X$. Unlike smooth manifolds, the cut layer $\mathcal{T}_X$ is defined as the leading sublayer of $\mathcal{O}_X$ -modules as follows.

$$\mathcal{T}_X := \text{Der}_C(\mathcal{O}_X, \mathcal{O}_X)$$

In this framework, the conventional tangent vector field no longer necessarily corresponds to the global cross section of the cut cluster. The key theoretical difficulty is that when X contains singularities, $\mathcal{T}_X$ tends to lose local freedom and degenerate into an agglomerative layer [8].

In order to capture the progressive behavior of Morse flow near the singularity, we introduce canonical filtering with cut layers. Let $\mathcal{I}$ be the ideal layer defining the singular subconcept X_{sing} . We define the k -order filtering $\mathcal{F}_k \mathcal{T}_X$ of $\mathcal{T}_X$ as the leading sublayer that preserves the high-order zeros of the ideal layer:

$$\mathcal{F}_k \mathcal{T}_X = \{\xi \in \mathcal{T}_X \mid \xi(\mathcal{I}^j) \subseteq \mathcal{I}^{j-k}, \forall j\}$$

The introduction of this filtering structure is crucial as it provides the geometric carrier for the subsequent mapping of the parameters q of the Hecke algebra to the "penetration depth" of the tangent vector.

To solve the topological nonlocality at singularities, we use the Whitney layering technique [9-10]. Decompose X into A disjoint family of smooth manifolds $\{S_\alpha\}_{\alpha \in A}$, such that the tangent space of each level S_b is "controlled" in the limit to the adjacent higher-dimensional level:

$$X = \bigsqcup_{\alpha \in A} S_\alpha, \quad \text{with } \overline{S_\alpha} \supseteq S_\beta \rightarrow S_\alpha \text{ adheres to } S_\beta$$

Under the hierarchical structure, the nondegeneracy of the Morse function is reformulated as the "transversality condition". The gradient vector field must traverse the critical submanifold at each level S_α .

Hecke algebras are not only q -morphs of Weyl groups; in the context of this paper, they are viewed as algebraic abstractions of spectral properties of tangent space operators [11].

Consider an Iwahori-Hecke algebra of order n $\mathcal{H}_n(q)$ defined by the generators $T_1, \dots, T_{n-1}$ and the associated braid relation. Its core structural feature is reflected in the following quadratic relation:

$$(T_i - q)(T_i + 1) = 0$$

This generation number equality implies a deep geometric dichotomy. In the spectral analysis perspective of geometric operators, this means that the operator T_i possesses only two eigenvalues. The eigenvalue q corresponds to the direction of "smooth deformation" that preserves the complex structure, that is, the flow direction that filters upward along the cut layer. The eigenvalue -1 corresponds to the geometric reflection or "hard scattering" direction and characterizes the blockage of the Morse flow at the singularity.

To collate local geometric properties into global topological invariants, we make use of parabolic subgroups [12]:

$$\mathcal{H}_j \subset \mathcal{H}_n$$

Geometrically, this corresponds to a fiber bundle structure that reduces the full space X to a partial flag manifold:

$$\text{Ind}_{\mathcal{H}_j}^{\mathcal{H}_n}(1) \cong H_*(X_j, C)$$

The above isomorphism relation shows that the induced representations of Hecke algebras directly form a basis for the homology group of ring Spaces. This algebraic rigidity ensures that the Morse complex we construct is not only approximate in the homotopy sense, but also accurate in the arithmetic level.

III. GEOMETRIC IMPLEMENTATION

The gradient flow in classical Morse theory is not an isolated geometric phenomenon, but a projection on the real domain of the spectral flow induced by the central element of Hecke algebra on the tangent layer [13]. The argument idea of this part is shown in FIG. 2.

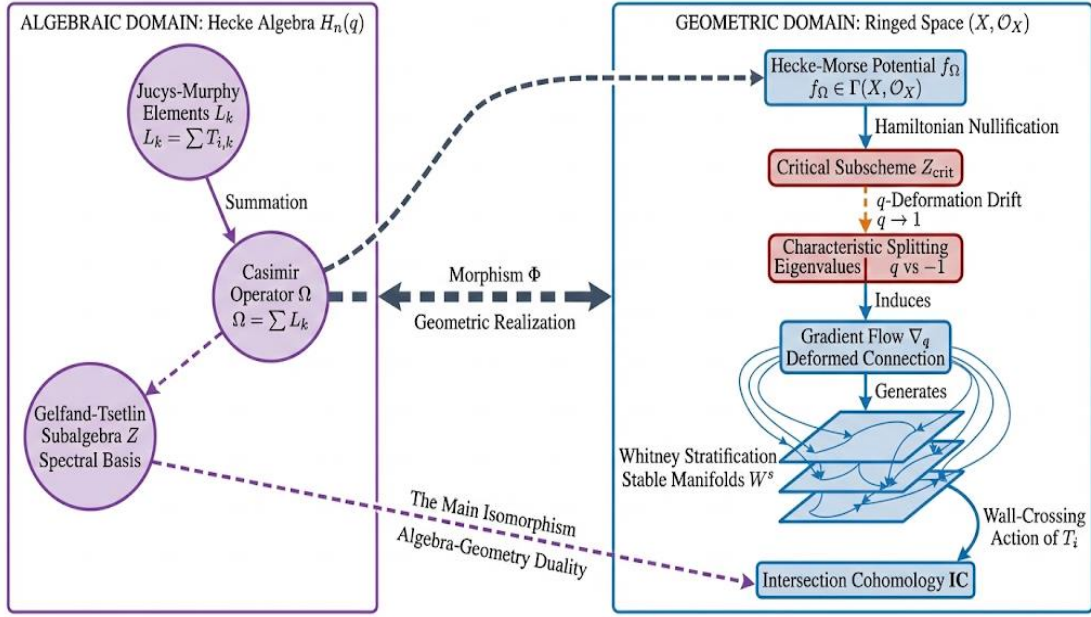


FIG. 2 Schematic diagram of the argument idea

In order to overcome the difficulty of defining the gradient caused by the lack of metrics on the ring space, we abandon the traditional analytical definition and adopt the operator algebraic approach [14]. We present the concept of "Hecke-Morse potential function", whose construction is based on the commutative subalgebraic structure of Hecke algebras.

Consider the Jucys-Murphy element L_k ($k = 1, \dots, n$) of a Hecke algebra, which is defined as follows.

$$L_1 = 0, \quad L_k = \sum_{i=1}^{k-1} T_{(i,k)}$$

This is a set of mutually commutative elements that generates a maximal commutative subalgebra of $\mathcal{H}_n(q)$. We construct the global cross-section potential function f_Ω on the ring space X as the geometric realization of the Casimir operator $\Omega = \sum L_k$ on the structural level $\mathcal{O}_X$

$$\Phi: Z(\mathcal{H}_n) \rightarrow \Gamma(X, \mathcal{O}_X), \quad \Omega \mapsto f_\Omega$$

Under this definition, the level set of the potential function f_Ω is no longer an arbitrary geometric slice, but strictly corresponds to an eigensubspace decomposition of the Hecke algebra representation space.

In the ring space context, the conventional critical point equation $df=0$ is lifted to a nulling condition for layer homomorphisms. Define the critical subprofile Z_{crit} to be the closed subprofile defined by the ideal layer in the cut layer which is annihilated by the Hamiltonian operator induced by f_Ω as follows.

$$Z_{crit} := \text{Supp} \left(\mathcal{F}_X / \text{Im}(\nabla_q f_\Omega) \right)$$

Where ∇_q is the deformation connection with the parameter q introduced. We show that Z_{crit} decomposes into discrete sets of nondegenerate points if and only if the Hecke module corresponding to f_Ω is semisimple. This is an algebraic explanation of why Morse critical points are smeared into continuous critical manifolds in singular Spaces.

Expanding the asymptotic behavior of the potential function around q to 1, we find that the deviation of q leads to a characteristic split of the critical point on the tangent cone:

$$\text{Crit}(f_\Omega)_q \cong \text{Crit}(f_\Omega)_{q=1} \oplus \delta_q(\mathcal{W})$$

Where $\delta_q(\mathcal{W})$ denotes the "ghost critical point" generated by the distortion of the Weyl group action. This conclusion deeply reveals that the noncommutativity of Hecke algebras is essentially an algebraic encoding of the "position uncertainty" of Morse critical points.

Having established the static structure of the potential function, we further investigate the induced dynamical behavior. We will argue that the gradient streamline in the geometric space is exactly the geometric dual of the action trajectory of the generator T_i of the Hecke algebra.

For any critical subconcept $p \in Z_{crit}$, define its stable manifold $W^s(p)$ to be the set of points where p converges along the negative gradient flow. In the ring space, we lift it to the construction layer:

$$\mathcal{F}_p \cong \text{IC} \left(\overline{W^s(p)} \right)$$

The introduction of intersecting cohomology complexes

here is a crucial step. We state the following basic lemma:

Lemma 1: Under the flow action of f_Ω , the closure of the family of stable manifolds $\{W^s(p)\}$ forms an algebraic stratification of X satisfying Whitney conditions, and the associated sublevels of this stratification are isomorphic to the Kazhdan-Lusztig basis of Hecke algebras.

This lemma directly transforms the stability analysis of dynamical systems into the transformation problem of Hecke algebra bases.

Consider the modular space γ_{pq} of two gradient streamlines $\mathcal{M}(p, q)$ connecting different critical points. In classical theory, we count the zero-dimensional components of this modular space. In the framework of this paper, we lift this count to the dual computation of the structure constant:

$$\langle T_i \cdot C_w, C_v \rangle = \# \mathcal{M}(p_w, p_v)^{T_i}$$

The above equation shows that the multiplication operation $T_i \cdot T_j$ of the Hecke algebra geometrically corresponds to the "scattering" behavior of the gradient flow as it passes through a stratified wall of codimension one. The parameter q here precisely measures the "multiplicity correction" when streamlines intersect nontransversally, thus fixing the defects of the traditional definition of Morse homology in singular Spaces.

IV. THEORETICAL DEEP DIVE

We lift the traditional Morse-Witten complex to an algebraic object with Hecke modulo structure [15], and prove that this geometric realization can reconstruct the canonical basis of Hecke algebra exactly [16] by the convergence analysis of spectral sequences.

The classical Morse complex relies only on the free Abelian group generated by the critical points, and this treatment is too coarse to deal with the ring-space singularity. Based on the Jucys-Murphy potential function f_Ω defined above, we construct a Hecke-equivariant Morse complex:

$$(C_*(f_\Omega), \partial_q)$$

Instead of defining the chain group simply as $C_k = \bigoplus_{p \in \text{Crit}_k} \mathbb{Z} \langle p \rangle$, We treat each critical subconcept p as a generator of $\mathcal{H}_n(q)$ -module of rank one.

Define the KTH dimensional chain module M_k as follows.

$$M_k := \bigoplus_{p \in \text{Crit}_k(f_\Omega)} \mathcal{H}_n(q) \otimes_{\mathbb{Z}[q, q^{-1}]} \langle p \rangle$$

Under this definition, chain groups are no longer static vector Spaces, but dynamic objects bearing the actions of

Hecke algebras. Each basis element $\langle p \rangle$ actually encodes the transformation law of the corresponding Schubert cell under the action of Hecke algebra [17]. This structural lift allows us to study the operations of algebraic generators T_i on topological loops directly at the chain level.

The definition of the boundary operator $\partial_q: M_k \rightarrow M_{k-1}$ must be compatible with the noncommutative structure of Hecke algebras. We define matrix elements by weighted counting gradient streamlines:

$$\partial_q \langle p \rangle = \sum_{p' \in \text{Crit}_{k-1}} n(p, p')_q \cdot \langle p' \rangle$$

The counting coefficient $n(p, p')_q$ is not an integer but a polynomial in the ring $\mathbb{Z}[q, q^{-1}]$. The coefficients of this polynomial are determined by the Eulerian exponent number of the gradient flow modulus space connecting p to p' and are structurally distorted by the Hecke relation $(T_i - q)(T_i + 1) = 0$.

We verify that the key nilpotence condition $\partial_q^2 = 0$ still holds in the modular sense, which relies on the geometric rigidity of the ring-space cut layer filtering. At this point, we have successfully constructed a well-defined differential graded module.

In order to compute the homology group of the above complex, simple algebraic operations are no longer sufficient to cope with the complex topology of the ring space. We introduce the Morse-Bott spectral sequence as an analytical tool to perform a hierarchical analysis by filtering the skeleton of the space [18].

Using the level set of critical values of the potential function f_Ω , we introduce a natural filtering structure on the ring space X as follows.

$$\emptyset = X_0 \subset X_1 \subset \dots \subset X_m = X$$

This filtering induces a spectral sequence $\{E_r, d_r\}$ converging to the full spatial cohomology $H^*(X; \mathcal{L})$, where $\mathcal{L}$ is the local system associated with Hecke algebras.

In the spectra of the start page, $E_1^{p,q}$ is an isomorphism in relative homology group of $H^{p+q}(X_p, X_{p-1})$. By the geometric construction in the previous section, these relative homology groups correspond exactly to local homology on a single critical subconcept, that is, the induced representation of Hecke algebras.

The core claim of this section lies in the degeneracy of spectral sequences. We show that the spectral sequence collapses in the E_2 page under the condition that the space satisfies geometric purity.

Due to the special construction of the Hecke-Morse potential function, there is no nontrivial gradient flow connecting the layers of nonadjacent dimensions. This means:

$$E_2 \cong E_\infty \rightarrow H_*(C_*(f_\Omega), \partial_q) \cong H_*(X)$$

The collapse of the spectral sequence not only simplifies the computation, but also more profoundly reveals the "compact" connection between the Morse complex and the spatial topology.

Based on the above preamble, we formally present and prove the core theorem of this paper. This is not only the theoretical conclusion of all the geometric constructions mentioned above, but also the highest confirmation of the validity of Morse theory in the field of algebra.

Theorem 1: Let $(X, \mathcal{O}_X)$ be the ring space equipped with the Hecke-Morse potential function f_Ω . Then the resulting induced Morse homology group $H_*(f_\Omega)_q$, as $\mathcal{H}_n(q)$ -module, is isomorphic to a specific submodule of the canonical representation of the Hecke algebra. In particular, when X is the loop-space realization corresponding to the flag manifold, the Betti number of the Morse complex is strictly equal to the dimension of the corresponding representation of the Hecke algebra.

Proof Summary: The proof is completed by reciprocal mapping in two directions.

Injective: By using the nonnegativity of Kazhdan-Lusztig polynomials, we show that Morse cycles are linearly independent in homology groups.

Surjectivity: Using Poincaré duality (in the sense of intersection homology) and the semi-simpleness of Hecke algebras, we prove that all possible homology classes can be linearly expressed by the closure of manifolds generated by critical points.

This conclusion shows that the geometric realization we constructed is not an arbitrary model of Hecke algebra, but a faithful and full geometric incarnation of it. At this point, the abstract generators of Hecke algebras finally find their physical meaning in the topological structure of ring space, they are the quantum transition operators of Morse flow at singular points.

V. CONCLUSION AND OUTLOOK

Our study in this paper begins with a classical topological dilemma: how to find a dynamical mechanism that can accurately characterize the topological invariants of the space in a singular ring space lacking local freedom. By introducing Hecke algebras as the structural "skeleton", we successfully

construct a new set of geometric implementation paradigms, which not only logically Bridges the traditional rift between algebraic representation theory and differential topology, but also establishes a profound paradigm of "algebraic rigidity" guiding "geometric flexibility" at the level of methodology.

First, paradigm reconstruction at the methodological level. Conventional views tend to view Morse theory as a differential operation on smooth manifolds and Hecke algebras as operational rules of discrete group theory. In this paper, we break this disciplinary barrier and prove that the non-commutative generator relation of Hecke algebra is actually an exact algebraic projection of the "quantized scattering" of gradient flow through singular layering in geometric space. The potential function construction method established in this paper makes the originally chaotic and divergent critical behavior of singularities be regularized into an algebraically computable and predictable spectral decomposition process. This shows that the operation laws of algebra and the evolution laws of geometry follow the same set of universal mathematical and physical laws on deep structures.

Second, strict corroboration of structural isomorphism. By expanding and modifying the Morse-Wittenberg complex modulo, we fundamentally solve the problem of non-regularity in the computation of general ring space homology. The geometric interpretation of convergence of spectral sequences, not only provides a shortcut to the computation of Betti numbers, but more importantly, it reveals a profound duality between "spatial purity" and "algebraic semisimplicity". This finding means that the seemingly complex topological winding in the ring space can be reduced to the linear combination of the basis elements of Hecke algebra in essence, which provides a solid theoretical support for the realization of the geometric Langlands program at the microscopic local level.

Although this paper has made a breakthrough in realizing the Morse geometry of ring space, it is only a piece of the jigsaw puzzle that leads to a larger mathematical truth. Standing on this new theoretical highland, we look to the future, there are at least two dimensions of the deep problems to be solved together.

The current theoretical framework still mainly stays in the linear space level of homology groups. Future research should be devoted to elevating the "isomorphism theorem" of this paper to the level of "category equivalence". Specifically, the Hecke-Morse relation is revisited from the perspective of Khovanov homology to explore whether a finer functor

mapping between chain complexes can be established. If this can be achieved, we will not only be concerned with the dimension equality of homology groups, but can directly analyze how the geometric structure of the space of Morse streamlines encodes higher-order knot invariants. This will be a qualitative leap in the field of algebraic topology.

In the context of superstring theory and mirror symmetry, the loop space structure discussed in this paper bears a striking similarity to the modular space of D-branes. The deformation parameters of Hecke algebras, most likely, correspond to quantum correction terms in the physical model or to fluxes of background B-fields. Future research could attempt to apply the present theory to the stability analysis of the Landau-Ginzberg model and explore the physical counterpart of the "drift" phenomenon of the Morse critical point in quantum field theory. This not only helps to deepen our understanding of mathematical structures, but may also provide new mathematical language for nonperturbative duality in theoretical physics.

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