Available online at www.rajournals.in

RA JOURNAL OF APPLIED RESEARCH

ISSN: 2394-6709

DOI:10.47191/rajar/v7i12.17

Volume: 07 Issue: 12 December-2021

International Open Access

Impact Factor- 7.036

Page no.- 2781-2785

Using Determinants for Determining Extreme Values of a Function of one Variable, Two Variables and Three Variables

Pham Ngoc Thinh

Truong Thịnh Education Center, Buon Ma Thuot City, Dak Lak Province, Viet Nam

ARTICLE INFO

Published Online: 28 December 2021

Corresponding Author: **Pham Ngoc Thinh**

ABSTRACT

Finding the maximum and minimum values of a function is essential in high school math. However, Vietnamese high school students have only been taught how to find the extreme values of a function of 1 variable. Seeing the extreme values of a function of 2 and 3 variables is a difficult problem for students. Using the determinants, our aim in this paper is to show the necessary and sufficient conditions for a continuous and differentiable function (1 variable, two variables, and three variables) to reach its maximum over a specified domain. Furthermore, our method can be used to find the extremes of n-variable differentiable functions.

KEYWORDS: Determinant of a matrix, Function, Relative maximization, Relative minimization.

INTRODUCTION

There are many valuable studies on the effective teaching of high school math in Vietnam. Some such results were recently published in the journal RESEARCH AND ANALYSIS JOURNAL OF APPLIED RESEARCH (see [1], [3], [4]). These are the results of research on math teaching topics. In this article, I will present the case of applying the determinant of a matrix to investigate the extremes of a function It consists of three parts. Part 1 discusses the definition of determinant, part 2 covers the extreme values of a function, part 3 deals with conclusions and discussions.

DEFINITION OF DETERMINANT

Let $A = [a_{ij}]$, i, j = 1, 2, ..., n is a square matrix of degree n, or n – matrix, with elements in filed K. The Leibniz formula for the determinant of an n-matrix A is related to expression. It is an expression involving the notion of permutations and their signature. A permutation of the set {1, 2, …, n} is a function δ that reorders this set of integers. The value in the i-th position after the reordering δ is denoted by δ_i. The set of all such permutations, the so-called symmetric group, is denoted S_n. The signature of δ is defined to be +1 whenever the reordering given by δ can be achieved by successively interchanging two entries an even number of times, and -1 whenever it can be achieved by an odd number of such interchanges. Given the matrix A and a permutation δ, the product $\prod_{i=1}^{n} a_{i,\delta i}$ is also written more briefly using Pi notation as $a_{1,\delta 1}a_{2,\delta_2}a_{3,\delta_3}....a_{n,\delta_n}$.

Using these notions, the definition of the determinant using the Leibniz formula is then the value

$$\det(A) = \sum_{\delta \in S_n}\left(\operatorname{sgn}(\delta)\prod_{i=1}^{n} a_{i,\delta i}\right)$$ a sum involving all permutations. Each summand is a product of entries of the matrix, multiplied with a sign depending on the permutation, called the determinant of matrix A.

See, a square matrix A of degree n is an element $(x_1,..,x_n)$ of vector space $(K^n)^n$, a matrix column A. Then $A \mapsto |A|$ is a map from $(K^n)^n$ to K.

After that, we let K is the real filed $\mathbb{R}$ or the complex filed $\mathbb{C}$.

Example: When n =1, the matrix of degree 1, A = [a], is elements in the filed; det(A) = a.

When n = 2, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$;

When n = 3,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh .$$

EXTREME VALUES OF A FUNCTION

1. Definition extreme values of a function:

Let f be a function with domain D and let c be a fixed constant in D. Then the output value f(c) is called

(i) The maximum relative value of f on D if and only if $f(x) \le f(c)$ for all x in D.

(ii) The relative minimum value of f on D if and only if $f(x) \ge f(c)$ for all x in D.

2. Determinant of degree 1 and extreme values of a function of one variable

Let f(x) be a real function of the real variable x for x in the domain D is a closed interval [a, b], and let us suppose that it possesses a convergent Taylor series of the form:

$$f(x) = f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \dots \quad (1)$$

around each point in the open interval (a, b).

Theorem 1: Let c be a stationary point of f(x), which is to say a point where f'(c) = 0

1. If $f''(c) > 0$ then f(x) has a relative minimum at x = c ;
2. If $f''(c) < 0$ then f(x) has a relative maximum at x = c.

In other words, if the matrix $[f''(c)]_{1\times 1}$ of degree 1 has a positive determinant, then f(x) has a relative minimum at x = c. If the matrix $[f''(c)]_{1\times 1}$ of degree 1 has a negative determinant, then f(x) has a relative minimum at x = c.

Proof:

Let be $f'(c) = 0$

we have $f''(c) = \lim\limits_{x \to c} \dfrac{f'(x) - f'(c)}{x - c}$

When $x \to c^-$ and $f''(c) > 0$ then $f'(x) - f'(c) > 0$ or $f'(x) > f'(c)$ hence f(x) has a relative minimum at x = c

When $x \to c^+$ and $f''(c) < 0$ then $f'(x) - f'(c) < 0$ or $f'(x) < f'(c)$ thus f(x) has a relative maximum at x = c

3. Determinant of degree 2 and extreme values of a function of two variables

For a function of two variables $f(x,y)$, defined over the rectangle $a_1 \le x \le b_1; a_2 \le x \le b_2$; and possessing a convergent Taylor series around each point (c_1, c_2) inside this region. Thus, $|x - c_1|; |y - c_2|$ sufficiently small, we have

$$f(x,y) = f(c_1,c_2) + (x-c_1)\frac{\partial f}{\partial c_1} + (y-c_2)\frac{\partial f}{\partial c_2} + \frac{(x-c_1)^2}{2}\frac{\partial^2 f}{\partial c_1^2} + (x-c_1)(y-c_2)\frac{\partial^2 f}{\partial c_1 \partial c_2} + \frac{(y-c_2)^2}{2}\frac{\partial^2 f}{\partial c_2^2} + \dots$$

Here

$$\frac{\partial f}{\partial c_1} = \frac{\partial f}{\partial x} \text{ at } x = c_1, y = c_2$$

And

$$\frac{\partial f}{\partial c_1} = \frac{\partial f}{\partial x} \text{ at } x = c_1, y = c_2$$

Let (c_1, c_2) be a stationary point of $f(x,y)$ which mean that we have the equations $\dfrac{\partial f}{\partial c_1} = 0; \dfrac{\partial f}{\partial c_2} = 0$

$$\text{We set } a = \frac{1}{2}\frac{\partial^2 f}{\partial c_1^2}; 2b = \frac{\partial^2 f}{\partial c_1 \partial c_2}; c = \frac{1}{2}\frac{\partial^2 f}{\partial c_2^2} \quad \text{And } u := x - c_1; v := y = c_2$$

$$\text{Hence } Q(x,y) = a(x-c_1)^2 + 2b(x-c_1)(y-c_2) + c(y-c_2)^2 = au^2 + 2buv + cv^2 = Q(u,v) \qquad (2)$$

Theorem 2: Let (c_1, c_2) be a stationary point of $f(x,y)$, which is a point where $f'(c_1, c_2) = 0$.

1. If $\begin{cases} a > 0 \\ \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0 \end{cases}$ then $f(x,y)$ has a relative minimum at (c_1, c_2).

2. If $\begin{cases} a < 0 \\ \begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0 \end{cases}$ then $f(x,y)$ has a relative maximum at (c_1, c_2).

Proof

Consider the homogeneous quadratic expression

$$Q(u,v) = au^2 + 2buv + cv^2 = a\left(u^2 + \frac{2b}{a}uv + \frac{b^2v^2}{a^2}\right) + cv^2 - \frac{b^2}{a}v^2 = a\left(u + \frac{bv}{a}\right)^2 + \left(c - \frac{b^2}{a}\right)v^2$$

Provided that $a \neq 0$

From below equation follows that $Q(u,v) > 0$ for all nontrivial u and v provided that

$$a > 0; c - \frac{b^2}{a} > 0 \Leftrightarrow a > 0; \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0 \qquad (3)$$

Similarly, $Q(u,v) > 0$ for all nontrivial, u and v provided that

$$a < 0; c - \frac{b^2}{a} < 0 \Leftrightarrow a < 0; \begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0 \qquad (4)$$

Conversely, if Q is positive for all nontrivial u and v, then two inequalities in (3) must hold, with a similar result holding for the case where Q is negative for all nontrivial u and v.

4. Determinant of degree 3 and extreme values of a function of three variables

Theorem 3: Let function of 3- variables

$$Q(x_1, x_2, x_3) := a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3 + a_{33}x_3^2 \text{ and } a_{11} \neq 0$$

1/ Necessary and sufficient conditions to form $Q(x_1, x_2, x_3) > 0$ are positive definite

$$\begin{cases} a_{11} > 0 \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0 \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0 \end{cases}$$

2/ Necessary and sufficient conditions to form $Q(x_1, x_2, x_3) > 0$ are negative definite

$$\begin{cases} a_{11} < 0 \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} < 0 \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} < 0 \end{cases}$$

Proof

Consider the homogeneous quadratic expression

$$Q(x_1, x_2, x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3 + a_{33}x_3^2 =$$

$$= a_{11}\left(x_1 + \frac{a_{12}x_2}{a_{11}} + \frac{a_{13}x_3}{a_{11}}\right)^2 + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)x_2^2 + 2\left(a_{23} - \frac{a_{12}a_{13}}{a_{11}}\right)x_2x_3 + \left(a_{33} - \frac{a_{13}^2}{a_{11}}\right)x_3^2$$

If $Q(x_1, x_2, x_3)$ is positive homogeneous quadratic expression, we choose x_1

$$x_1 + \frac{a_{12}x_2 + a_{13}x_3}{a_{11}} = 0$$

We have the homogeneous quadratic expression of two variables

$$P(x_2, x_3) = \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)x_2^2 + 2\left(a_{23} - \frac{a_{12}a_{13}}{a_{11}}\right)x_2x_3 + \left(a_{33} - \frac{a_{13}^2}{a_{11}}\right)x_3^2$$

So $P(x_1, x_2)$ is the positive homogeneous quadratic expression that

$$\begin{cases} a_{11} > 0 \\ a_{22} - \dfrac{a_{12}^2}{a_{11}} > 0 \\ \begin{vmatrix} a_{22} - \dfrac{a_{12}^2}{a_{11}} & a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} \\ a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} & a_{33} - \dfrac{a_{13}^2}{a_{11}} \end{vmatrix} > 0 \end{cases}$$

We consider

$$D := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \xrightarrow[h_3 \to \frac{a_{13}}{a_{11}} h_1 - h_3]{h_2 \to \frac{a_{12}}{a_{11}} h_1 - h_2} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \dfrac{a_{12}^2}{a_{11}} & a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} \\ 0 & a_{23} - \dfrac{a_{13}a_{12}}{a_{11}} & a_{33} - \dfrac{a_{13}^2}{a_{11}} \end{vmatrix}$$

Hence

$$D = a_{11} \begin{vmatrix} a_{22} - \dfrac{a_{12}^2}{a_{11}} & a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} \\ a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} & a_{33} - \dfrac{a_{13}^2}{a_{11}} \end{vmatrix}$$

Show that necessary and sufficient conditions for D > 0 are

$$\begin{vmatrix} a_{22} - \dfrac{a_{12}^2}{a_{11}} & a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} \\ a_{23} - \dfrac{a_{12}a_{13}}{a_{11}} & a_{33} - \dfrac{a_{13}^2}{a_{11}} \end{vmatrix} > 0 \text{ so } \begin{cases} a_{11} > 0 \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0 \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0 \end{cases}.$$

Example

Find the maximization – minimization of function $f(x,y) = x^3 + y^3 - 3xy$ at a stationary point

We have

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 - 3y = 0 \\ \dfrac{\partial f}{\partial y} = 3y^2 - 3x = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x = x^4 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = 0 \\ y = 0 \end{cases} \\ \begin{cases} x = 1 \\ y = 1 \end{cases} \end{array} \right.$$

So two stationary points are $M_1(1,1)$ or $M_2(0,0)$

Consider at $M_1(1,1)$

We set $a = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial c_1^2} = 3; b = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial c_1 \partial c_2} = -\dfrac{3}{2}; c = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial c_2^2} = 3$

So determinant $\begin{vmatrix} 3 & -\dfrac{3}{2} \\ -\dfrac{3}{2} & 3 \end{vmatrix} = \dfrac{27}{4} > 0$ that the function had minimization at $M_1(1,1)$

We consider at $M_2(0,0)$

We set $a = \frac{1}{2}\frac{\partial^2 f}{\partial c_1^{\,2}} = 0; b = \frac{1}{2}\frac{\partial^2 f}{\partial c_1 \partial c_2} = -\frac{3}{2}; c = \frac{1}{2}\frac{\partial^2 f}{\partial c_2^{\,2}} = 0$

So determinant $\begin{vmatrix} 0 & -\frac{3}{2} \\ -\frac{3}{2} & 0 \end{vmatrix} = -\frac{9}{4} < 0$ and $a = 0$ that the function is not maximization and minimization at $M_2(0,0)$

CONCLUSIONS AND DISCUSSIONS

We used the determinant criteria for presenting the extreme values of a function of one variable, two, and three. We say the theorems of the result, and we give detailed proofs. On the contrary, the reference materials are generally said but not demonstrated in detail. Moreover, by applying the method of this paper, we can find the extreme values of some functions of n- variables.

ACKNOWLEDGMENTS

I want to take this opportunity to express my gratitude to the Organizing Committee of ICREAMS 2015, Roi-Et Rajabhat University, Thailand, who kindly invited me to participate at this Conference; to my Colleagues for their great exciting discussions and comments about my paper. I want to express our gratitude to Assoc. Prof. Dr. Nguyen Huynh Phan for his valuable suggestions and discussions.

REFERENCES

1. Cu Thi Kim Dung and Nguyen Thi Thu Hien and Nguyen Thi Thu Thuy: Topological Essence of the Concept “Limit of a Function” in the General Mathematics. RESEARCH AND ANALYSIS JOURNAL OF APPLIED RESEARCH, ISSN: 2394-6709 DOI:10.47191/rajar/v7i12.07, Volume: 07 Issue: 12 December-2021; International Open Access, Impact Factor- 7.036 Page no.- 2741-2744.
2. Richard Bellman (1969), Introduction to Matrix Analysis, Society for Industrial and Applied Mathematics Philadelphia.
3. Vu Thi Binh: Improve the Ability to Use Mathematical Representation for Junior High School Students in Vietnam. RESEARCH AND ANALYSIS JOURNAL OF APPLIED RESEARCH, ISSN: 2394-6709, DOI:10.47191/rajar/v7i7.03, Volume: 07 Issue: 07 July-2021; Impact Factor- 7.036; Page no. 2443-2448.
4. Le Trong Toan and Do Thi Thanh Thao: Topological Essence of the Concept “Limit of a Numerical Sequence” in the General Mathematics. RESEARCH AND ANALYSIS JOURNAL OF APPLIED RESEARCH, ISSN: 2394-6709, DOI:10.47191/rajar/v7i12.08,Volume: 07 Issue: 12 December-2021, International, Open Access, Impact Factor- 7.036, Page no.- 2745-2748.
5. Ngo Viet Trung (2001), Linear Algebra, Publisher Ha Noi National University.