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Topological Essence of the Concept "Limit of a Function" in the General Mathematics

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ABSTRACT

The limit of a function is one of the most important concepts in high school mathematics. But unfortunately, not many students understand the essence of this concept. They had to accept it. This paper presents the topological essence of a limit of a function and discusses methods for teaching this concept in general education in Viet Nam. It consists of four parts. Part 1 talks about the definition of "Limit of a function", part 2 covers the Mathematical essence of the concept limit of a function, part 3 deals with Defining the limit of a function in a topological space of real numbers and part 4 presents Adding a topology on the set of real numbers and limit of a function. In each section, we include comments to help teach these notions and concepts.

KEYWORDS: Mathematics; Limit; function; Interval; union; Intersection, Topology.

1. LIMIT OF A FUNCTION AND SOME COMMENTS.

The concept of limit of a function is taught in the math of grade 11 in Vietnam (the universal education program of Vietnam lasted 12 years), and it is presented as follows:

1.1.Definition : *Given the function f(x) definite on the set of numbers $(a - \Delta, a) \cup (a, a + \Delta)$ with $\Delta > 0$. We say that f(x) has a finite limit of s when x gradually moves to a and symbols* $\lim_{x \to a} f(x) = s$ *if, with every arbitrary positive number ε, there will be a positive real number δ such that for all x where* $0 < |x - a| < \delta$, *then* $|f(x) - s| < \varepsilon$.

It can be said that most students do not understand that why the limit of a function is defined as such, thus for the whole student life, they accept it without knowing why. So how to help students understand this important mathematical concept.

Mathematics is the way that humans think of to reflect existence. So what does the concept of the limit of a function reflects in life?

Let's start from the following paradox, called the "paradox that Rabbit fails to catch up with Turtle": Rabbits run slower than tortoise?. Because rabbit runs faster than turtle (assuming 100 times faster), rabbit stands at A and lets turtle stand at B, in front of him a distance a. Both of them start at the same time. See what happens then. When the rabbit runs to B, the turtle moves forward one distance $\frac{a}{100}$; this position is denoted as B_1. When the rabbit runs to B_1, the turtle moves forward by a space ($\frac{a}{100}$): 100 = $\frac{a}{100^2}$, this position is designated as B_2. When the rabbit reaches B_2, the turtle moves forward by a distance ($\frac{a}{100^2}$): 100 = $\frac{a}{100^3}$. The process continues so, and we see that the turtle is always in front of the rabbit a distance $\frac{a}{100^n}$. When n is excellent enough, $\frac{a}{100^n}$ it will be very little, but it is always not 0. Therefore, the rabbit can not catch up with the turtle.

All arguments are not wrong, but it is not acceptable in our lives. So what does the conflict come from? In this case, we only use four arithmetic operations to reflect a life phenomenon rabbit chasing turtle. We can conclude that the four essential arithmetic operations, addition, subtraction, multiplication, division, are not sufficient to reflect phenomena of life. Therefore, we should use other mathematical concepts to reflect this existence. The idea allows the conception "number in the form $\frac{a}{100^n}$ with n is great enough, and it is considered zero" (remember that the rabbit caught up with the turtle). But this is a saying, while math needs exact concepts and calculations.

We, at this moment, present the exact mathematical concept about the above saying. This is also the way that helps students to understand the idea "Limit of a function."

Recalling that ℝ is denoted a set of real numbers; An open interval (or an interval) with the ends a, b are the set of numbers

(a, b): = {x ∈; ℝ a <x <b}.

So the number d: $= \frac{a+b}{2}$ is the midpoint of the interval (a, b), and from d to two ends a and b, the length is $\frac{b-a}{2}$.

Symbol *δ*: $= \frac{b-a}{2}$.

We have

(a, b) = {x; $|x - d| < \delta$}.

From this feature was seen that to have an interval (a, b), we just let a positive number δ as the length of the half of that interval, and the number d is the midpoint of this interval. When the number d is already known, entirely determined interval by the positive number *δ*. Therefore, we will sign interval (a, b) by:

(a, b) = d(δ).

From here, we see the phrase *"with any arbitrary positive number ε, and there exists a positive real number δ such that for all x where 0 <*$|x - a|$ *< δ, then* $|f(x) - s|$ *< ε"* is essentially to say: "Every interval containing the number s will contain the values of the function f(x) with x running in the interval a(δ) with some δ."

1.2. Proposition: *Given the function f(x) defined on the set of numbers (a - Δ, a)* ∪ *(a, a + Δ) with Δ > 0. Then* $\lim_{x\to a} f(x) = s$ *if and only if all intervals containing the number s will contain the function f(x) values with x running in a certain interval.*

Proof: Indeed, the statement: [every arbitrary positive number ε, then there exists a positive real number δ such that for all x where 0 <$|x - a|$ < δ, then $|f(x) - s < \varepsilon$] is if and only if when arbitrary interval s (ε) containing number s, then there exists an interval a (δ) containing number a such that all x in this interval, then f (x) in the interval s (ε).

Symbol Imf D is the value set of function f(x) with a specific defined domain D.

1.3.Proposition: $\lim_{x\to a} f(x) = s$ *if and only if all interval containing s contain Imf u (δ) with a specific range u (δ).*

Proof: Indeed, the statement: [arbitary interval s (ε) containing number s, then there exists an interval a (δ) containing number a such that all x in this interval, then f (x) in the interval s (ε)] is equivalence with the Proposition.

To avoid repeating, from now on, we always consider the function f(x) defined on the set of numbers (a - Δ, a) ∪ (a, a + Δ) with Δ > 0.

2. MATHEMATICAL ESSENCE OF THE CONCEPT LIMIT OF A FUNCTION

The concept limit of a function brings deep mathematical meaning, and it is formed on a different fundamental concept of mathematics, the topology concept appeared as follows.

On the set of real numbers ℝ, we consider a family ℱ of subsets A of ℝ such that each A is a union of arbitrary numbers of interval in the form (a, b) above. Thus,

ℱ = {A ⊆ ℝ; $A = \bigcup_{i\in I}(a_i, b_i)$}; ($a_i$, b_i) are intervals}.

2.1. Theorem: $\lim_{x\to a} f(x) = s$ *if and only if all elements A of the family ℱ containing s contain Imf u(δ) with some intervals u(δ).*

Proof: Because a subset A is a union of intervals, therefore the statement : [all elements A of the family ℱ containing s contain Imf u(δ) with some intervals u(δ)] is equivalence the following statement: [arbitary interval s (ε) containing number s, then there exists an interval a (δ) containing number a such that all x in this interval, then f (x) in the interval s (ε)].

2.2. Comment: *The family ℱ acts as a ruler to measure the closeness of a number s with the set of values of the function f(x). More precisely, we say that the function f(x) gradually goes to the number s when the variable x moves towards a if all elements of the family ℱ containing the number s contain Imf a(δ) with a certain δ.*

2.3. Theorem: *The family* ℱ = {A ⊆ ℝ; $A = \bigcup_{i\in I}(a_i, b_i)$} has 3 foundation properties as follows:

1/ The empty set ∅ ∈ ℱ and the set ℝ ∈ ℱ;

2/ The family ℱ is closed with arbitrary union, i.e. with all subsets A_j ∈ ℱ, then $\bigcup_{i\in J} A_j \in$ ℱ.

3/ The family ℱ is closed with finite intersection, which means that with a finite number of subsets $A_1, A_2, \ldots A_k$ ∈ ℱ, then $\bigcup_{t=1}^{k} A_t \in$ ℱ.

Proof :

1/ Because the interval (a, a) = ∅ and ℝ = $\bigcup_{n=0}^{\infty}(-n, n)$; n are natural numbers. Therefore, it is clear that ∅ and ℝ are in ℱ

2/ Now let $\{A_j\}_{j\in I}$ be some collection of elements in ℱ which we can

assume to be nonempty. If p ∈ $\cup_{j\in I} A_j$ we can choose some j ∈ I, and there exists an open ball B ⊆ A_j containing p. But B ⊆ $\cup_{j\in I} A_j$ so this shows that

$\cup_{j\in I} A_j \in \mathscr{F}$.

3/ The first we see that

$$(A\bigcup B)\bigcap(C\bigcup D)=(A\bigcap C)\bigcup(A\bigcap D)\bigcup(B\bigcap C)\bigcup(B\bigcap D)$$

Equivalently,

$$\left(\bigcup_{i\in I}(a_i, b_i)\right)\bigcap\left(\bigcup_{j\in J}(c_j, d_j)\right)=\bigcup_{i\in I, j\in J}\left[(a_i, b_i)\bigcap(c_j, d_j)\right]$$

But the intersection of two intervals is an interval, hence, $A_1, A_2 \in F$ then $A_1 \cap A_2 \in F$.

From this, we have if $A_1, A_2, \ldots A_k \in \mathscr{F}$, then the intersection $A_1 \cap \cdots \cap A_k \in \mathscr{F}$.

2.4. Defining topology on the set of real numbers ℝ: A family ζ of subsets B of ℝ is called a topology on ℝ if the family ζ satisfies three following conditions:

1/ The empty set ∅ ∈ ζ and the set ℝ ∈ ζ;

2/ The family ζ is closed with the arbitrary union, i.e., with all subsets $B_j \in \zeta$, then $\bigcup_{i\in J} B_j \in \zeta$.

3/ The family ζ is closed with finite intersection, which means that with the finite number of subsets $A_1, A_2, \ldots A_k \in \zeta$, then $\bigcup_{t=1}^{k} B_t \in \zeta$.

2.5. Corollary: Family $\mathscr{F}$ above is a topology on ℝ.

2.6. Example: Family ζ = {∅, ℝ} (only two elements are ∅ and ℝ) is a topology on ℝ, it is called crude topology.

2.7. Example: Family ζ = {all subsets of ℝ} is a topology on ℝ, it called discrete topology.

3. DEFINING LIMIT OF A FUNCTION IN A TOPOLOGICAL SPACE OF REAL NUMBERS ℝ

3.1. Definition: Suppose that on the set of real numbers ℝ, given a certain topology ζ. We say the function f(x) has a limit s or moves gradually towards s when the variable x moves gradually towards a if every element of ζ, which contains s, contains Imf a(δ) with a certain δ.

So the essence of the concept limit of a function is the limit in a specific topology on the set of real numbers ℝ.

3.2. Comment: If ζ is a crude topology, any function f(x) has a limit as any number, i.e., every function f(x) will move gradually towards any number.

3.3. Comment: If ζ is the discrete topology, a function f(x) has a limit of s if and only if f(x) is a constant function s, i.e. f(x) = s with all x.

3.4. Comment: The limit of function f(x) is defined as above is the limit of the function f(x) with $\mathscr{F}$ topology.

3.5. Proposition: If on ℝ, given $\mathscr{F}$ topology and function f(x) has a limit, this limit is unique.

Proof: Suppose that f: A → ℝ and c ∈ ℝ is an accumulation point of A ⊂ ℝ.

Assume that

$$\lim_{x\to c} f(x) = L_1;$$

$$\lim_{x\to c} f(x) = L_2$$

where $L_1, L_2 \in$ ℝ. For every $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$0 < |x - c| < \delta_1$ and x ∈ A implies that $|f(x) - L_1| < \varepsilon/2$,

$0 < |x - c| < \delta_2$ and x ∈ A implies that $|f(x) - L_2| < \varepsilon/2$.

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, since c is an accumulation point of A, there exists

x ∈ A such that $0 < |x - c| < \delta$. It follows that

$$|L_1 - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \varepsilon .$$

Since this holds for arbitrary $\varepsilon > 0$, we must have $L_1 = L_2$.

4. ADDING A TOPOLOGY ON THE SET OF REAL NUMBERS AND LIMIT OF A FUNCTION

4.1. Definition: *Let any positive number ε. We say that the interval (a, b) is ε- interval if its length is 2ε.*

We will build a topology on ℝ as follows:

Put

$\mathscr{M}$: = {C ⊂ ℝ; with all c ∈ C, there exists ε- in the interval (a, b) such that c ∈ (a, b) ⊂ C}.

4.2. Proposition: Family $\mathscr{M}$ is a topology on ℝ coinciding with the topology $\mathscr{F}$, i.e., $\mathscr{M} = \mathscr{F}$ (the two sets $\mathscr{M}$ and $\mathscr{F}$ are equal).

Proof: We have C ∈ $\mathscr{M}$ if and only if c ∈ C then C contains an interval containing c, that means if and only if C is a union of some intervals, therefore if and only if C ∈ $\mathscr{F}$.

4.3. Comment and Conclusion: The limit of function stated in the mathematical textbook of class 11 in Vietnam is the limit of a function for $\mathscr{M}$ topology and thus also the limit for $\mathscr{F}$ topology. But topological concepts are not taught in secondary mathematics, so students do not understand the limit of a function. Hence we need to use the methods suggested above to teach this concept to students. They are both intuitive and easy to understand.

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