Analysis of Prey Predator Interaction with Harvesting System on Mathematical Modelling

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ARTICLE INFO	ABSTRACT
Published Online: 21 July 2023 **Corresponding Author:** A. N. Mohamad	In the Modern World, everything will be systematically developed by using best modelling. In our article we will prepare and analyzed in the area of Dynamics of System of Two Prey and One Predator by using Holling Type – II Functional Response, and we designed the model such as First Prey and the Second prey a ratio – dependent response, where harvesting of each prey species is taken into consideration. And also, the model is used to study the Ecological Dynamics of the Fox – Antelope – Rabit (FAR) in a given habitat. We also focus the effect of harvesting on prey species. We discussed Local and Global Stability Analysis of the system were carried out, and also we analyze numerical simulation for particular variables and then, we conclude and show that the Result of Analysis of our model that the Three Species would co – exist if the Antelope and Rabbit were not harvesting beyond their Intrinsic Growth Rate (IGR).

1. INTRODUCTION

1.1. Basic concepts and relevant Definitions for our article.

In our Real – World Systems are mostly complexity and have number of Interrelated Components are involved, among the various interrelated components the proposed this article will be focused on predator and prey interaction with prey harvesting. For the dynamic relationship between predator and their prey has long been and will continue to be one of the dominating themes in both colony ad Mathematical Ecology due to UniversalExistence and Importance. An important ubiquitous problem co – existing (or persistence) of species (Manju and Rachana – 2012); Again (Abrams – 1996) studied a prey – predator Dynamics, where the predator species partially depends up on the prey species in a two-bath habitat and obtained the conditions for asymptotic stability.

1.1.1. Functional Response

A functional response is described as predators instantaneous per capita feeding rate as a function of prey (Hulling – 1959). This means that the consumptions rate of an individual predator depends on the prey density. Understanding and clearly qualifying functional response is at the heart of the Ecological Modelling.

According to Abrams and Ginzburg – 2000, functional responses are generally categorized as:

Prey – Density Dependent Response denoted by $f(m)$ and Ratio – Dependent as $f(m/p)$ and Prey – Predator Density Dependent as $f(m,p)$ where m,p represent prey -predator population species respectively. In 1959 – Hulling introduced three types name as Holling Type – I, II and III, who categorized the prey – density responses.

a. Holling Type – I

Hulling Type – I, functional response which is the standard mass action or linear response as

$$f(m) = am \qquad (1)$$

$Where, a > 0$, it is the attack rate of the predator. Holling Type – I's response is found in **Passive Predators** like spiders. The number of flies caught in the net is proportional to fly density. Prey mortality due to predation is constant.

b. Holling Type – II

Hulling Type – II, it is also called the **Cyrtid Functional** response and it is represented by the function

$$f(m) = \frac{bm}{1+cm} \qquad (2)$$

$Where, b \ and \ c$, they are positive constants that describe the effects of capture rate and handling time on the feeding rate of the predator (Skalski and Gillian – 2001). The Holling

Type – II response is the most common type of functional response and is well documented. According to (Sharov – 1996), at low prey densities, the predator spends more than in searching the prey while at high prey densities, the predator spends more time handling the prey. Relating this t equation (2), we know that, at $\begin{cases} at\ Low\ Prey\ Densities\ \ \ b > c \\ at\ High\ Prey\ Densities\ \ c > b \end{cases}$

Either way, the number of prey that a predator can consume is limited or consequently the predator reaches a saturation level. for example:

Small mammals destroy most of gypsy moth pupae in sparse populations of gypsy moth.

However, in High – Density defoliating populations, small mammals kill a negligible proportion of pupae (Sharov – 1996)

c. Hulling Type – III

Hulling Type – III, functional response is represented by the function below, that is,

$$f(m) = \frac{dm^x}{F + m^x}$$

$$(3)$$

Where $x > 1$, it is the **Encounter Rate** between predator and prey before predator reaches maximum efficiency and also $d\ and\ F > 0$

According to (Sharov – 1996), Holling III functional response occurs in predators which increase their search activity with increasing prey density.

1.2. For example

Many predators respond to kairomones (Chemical emitted by prey) and increase their activity; Polyphagous vertebrate (e.g., Birds) can switch to the most abundant prey species by learning to recognize itvisuality, Mortality first increases with prey increasing density, and then declines. If predator density is constant (e.g., Birds, Small mammals) then they can regulate prey density only if they have a Holling Type – III functional response, because this is the only type of functional response for which prey mortality can increase with increasing prey density. However, regulating effect of predators is limited to the interval of prey density where mortality increases. If prey density exceeds the threshold value of this interval, then mortality due to predation starts declining, and production will cause a Positive – Feed – Back/ As a result, the (Decreases or Food Shortage) will stop their reproduction.

1.3. To Find Ratio Dependency – (m/p)in the Holling Type – II - Function

It is obtained by substituting the Prey – Predator ratio(m/p) for prey density (m) in the Holling Type – II function is

$$f(m/p) = \frac{d(m/p)}{n + (m/p)} = \frac{dm}{np + m} \qquad (4)$$

$Where, d > 0\ and\ m > 0$

That stand for capture rate and half – saturation constant for predator p respectively (Xiao and Ruan – 2001). (Bedding ton– 1975) and (De Angeles – 1975) et. al. proposed

independently amore general formof ration – dependent function response, that is,

$$f(m/p) = \frac{am}{1 + bm + c(p-1)} \qquad (5)$$

$Where, a > 0$, it is the attack rate of the predator, and $Where, b > 0$, it is positive constant that describe the effects of capture rate and c describes the magnitude of interference among predators. Thus, this functional response takes into account the delay in time incurred by the predators as a result of inter specific competition for the different prey species. For Prey – Predator – Dependency, the consumption rate of the predator depends on both the prey and predator density.

For examples of Prey – Predator – Dependency Functional Response are

$\begin{cases} a.\ Crowly - Martin\ Model \\ b.\ Hassel\ Varley\ Model \end{cases}$

Both documented in (Skalski and Gillian – 2001)

For Crowley – Martin Model is given by

$$f(m/p) = \frac{am}{1 + bm + c(p-1) + bcm(p-1)}$$

$$(6)$$

1.4. Remark:

An important distinction between the Bedding ton – DeAngelis Model and the Crowly – Martin Model is that the effects of predator interference (Competition/Infighting among predators) on the feeding rate becomes negligible under conditions of high prey density while the latter assumes that the interference remains important even at high prey density (Skalski and Gillian – 2001).

This can be seen by letting $m(t) \rightarrow \infty$ in both models. It is noticed that the Bedding ton – DeAngelis Model gives an expression

$$\lim_{m \to \infty} f(m/p) = \frac{a}{b} \qquad (7)$$

Independent of predator interference parameter c and while the Crowly – Martin Model give an expression

$$\lim_{m \to \infty} f(m/p) = \frac{a}{b + bc(p-1)} \qquad (8)$$

Dependent on predator interference and density

According to (Abrams and Matsuda – 1996), generally, Ratio – Dependent type of functional response is more appropriate in Predator – Prey interactions in which, the predator takes too much time in handling the prey, aggression and/or competition among predators is time consuming which prolongs searching time, the prey adopt an anti – predator behavior and try to evade the predator. In the proposed study, we intend to use Holling Type – II function response, ratio – dependent response and harvesting effort of prey species with second prey adopts anti – predator behavior.

1.5. Aim and Achievement

Mathematical Ecology requires study of populations that interact there by affecting each other's growth rates. So that, many scientists have carried out different studies on the predator – prey interaction. Our main of this article is to understand the dynamics of interaction between one predator

and two preys with harvesting of prey species using Holling Type – II functional response and ratio dependent response. Therefore, in this research, we modify a mathematical model which clearly describes the existing reality between two preys and one predator system with constant harvesting of prey in maintaining both populations in stable ecological system, and hence we will raise the following research-oriented questions

1. Can we modify a mathematical model which efficiently describes the dynamics of two preys and one predator with constant harvesting effort of prey species?
2. Are the systems both locally and globally stable?
3. What are the impacts of constant harvesting effort of prey species on the local stability of the predator prey dynamics?
4. Is there any biological hindering factor on the coexistence of prey and predator dynamical system with constant harvesting effort of prey?
5. Find out how harvesting hunting of the preys impacts on the long term stability of the ecosystem so as to establish conditions for optimal harvesting of the prey?
6. Analyze local and global stability of the predator – prey dynamics.
7. Give biological analysis on the coexistence of two prey and predator dynamical system with harvesting of prey.

1.6. Signification of the Research

We hope that, if the formulated mathematical model clearly reflects the effect of constant harvesting of preys on stable co – existence of these population. The findings of this research will be benefit different bodies differently. Hence, it will

a. Help policy makers in making decisions to protect endemic species in the world.
b. Initiate other researchers to undertake further extension and precise mathematical analysis.
c. Provide details information how to manage the long term stable co – existence of the predator and two prey population of the authorized bodies.

In all situations we are using a system of Ordinary Differential Equations. Then we studied the qualitative behavior of the formulated mathematical model governing the system. Its local and global stability were discussed by using Jacobean Matrix and Lapunove Functions respectively. The analytical results tested through numerical simulation using MATLAB software.

2. LITERATURE REVIEW

Mathematical Modeling of exploitation of biological resources is still a very interesting field of research/ Systematic mathematical analysis can often lead to better understanding of bio – economic models. In the last few decades, interest has growing steadily in the designing and studying of mathematical models of population interactions. Mathematical modelling and analysis of ecological problems was first done by (Volterra – 1927). He had been introduced

to an ecological problem that in the years after the First World War, the population of the predatory fishes caught in Upper Adriatic Sea was found to be considerably higher than in the years before the war. Whereas the proportion of prey fishes was down: In order to come out with an explanation to this ecological problem, Volterra formulated and analyzed a system of ordinary differential equations which is represented as:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dy) \end{cases} \quad (2.1.)$$

where x and y , they were the densities of the prey and predator fish respectively. Thos system of differential equations was also studied independently by (Lotka – 1925) in the context of chemical kinetics and is known as the Lotka – Volterra Model (LVM). Volterra study showed that the steady state for the co – existence of the prey fish and predatory fish was periodic and that a pause of fishery world indeed led to an increase of the predators and a decrease in the prey. Then, many ecological models have been formulated and analyzed to study various phenomena. (Freedman – 1980) came up with generalized Pre – Predator Model represented by the system of equations as shown below:

$$\begin{cases} \frac{dx}{dt} = xg(x) = yf(x) \\ \frac{dy}{dt} = y[-c + p(x)] \end{cases} \quad (2.2.)$$

where x and y , they were the densities of the prey and predator respectively, $g(x)$, it is the grown rate of the prey in absence of the predators, $f(x)$, it is the functional response of the predators with respect to prey x and $p(x)$, it is the numerical response of the predator. In most cases, $p(x)$, it is a product of a constant and $f(x)$. The functions $f(x)$ and $p(x)$, they are continuous and differentiable functions.

2.1. Models for Two Preus and One Predator System with Harvesting

(Kar – 2003) studied a prey – predator system with delay, Holling Type – II functional response and harvesting of the prey. The study showed that as harvesting effort increased, the predator's population decrease as expected. More importantly, if the harvesting effort was above a critical value which was determined in the study, the dynamic system changed from limit cycle to Globally Asymptotic Stability (GAS). This showed that harvesting of the prey alone indirectly affected the population density of the predators and also played a crucial role in stabilizing the dynamics of the prey – predator systems. The delay term was included to ensure that only mature preys were harvesting.

(Chaudhri and Kar – 2004) proposed and analyzed a fishery models of tow preys and one predator system in which the prey was being harvested and the feeding rate of the predator increases linearly with prey density. They derived conditions for Global Stability (G. S) of the system, using a Lapunove Functions. Using Ponryagins Maximal Principal, they

established the conditions for Optimal Harvest. However, the prey dependent – Linear Functional Response used in their models don't represent the feeding patterns of most species as compared to Holling Type – II response or Ratio – Dependent Response. (Greem – 2004) studied a model on two preys and one predator system in which competing and predation followed the density gradient of the prey. It was discovered that when the predator divides its time between the two preys depending on their comparative density, the predator stabilized the system. Limit cycle and chaotic behavior in the system were also investigated. The model did not consider competition among prey and prey harvesting. (Vlastmil and Eisner – 2006) studied a one consumer and two resources population dynamics systems in which the resources was spatially distributed between two patches. The studied showed that the resources grow exponentially, handling times are zero and apparent competition always leads to extinction of the weaker resource. However, with logistic growth and Holling Type – II functional response included in the mode, species permanence was guaranteed. This showed the importance of incorporating logistic growth in prey – predator models.

2.2. Models Incorporating Ratio – Dependent Response

(Dubey and Upadhyay – 2004) studied a model on two predators and prey system with ration to be – dependent predators' growth. Their results showed that the role of food conversion coefficients of predators in ratio – dependent models were crucial in determining the stability behavior of planer – equilibrium. They derived sufficient conditions for the system to be uniformly persistent. Conditions were derived for the co – existence equilibrium to be (GAS). The rate food conversion coefficients of predators in stabilizing the model are also compared to that of the effect of harvesting. The model proposed for this research incorporates both the Holling Type – II and Ratio – Dependent responses, something none of the models above have done, The justification for our model is that it seeks to capture the dynamics of predator feeding in two preys. One prey is easy to capture and the predator takes less time in searching and handling it, this behavior is modeled by the Holling Type – II response. The second prey is hard to capture and the predator takes much time in searching and handling the prey, this behavior is modeled by the Ratio – Dependent Response. They preys are harvested and this has an impact on the stability of the ecosystem. It is noted, that in the Bedding ton – De Angelis Ratio – Dependent Response in equation (1.5) that is,

$$(m/p) = \frac{am}{1+bm+c(p-1)} \qquad (5)$$

Parameter $c > 0$ describes the magnitude of interference among predators. However, the model proposed here has only one predator species and as such interference among predators is not considered. Therefore, Parameter $c > 0$ it is replaced with $d_2 > 0$ to represent the magnitude of Anti –

Predators behavior by the hard to capture prey. Also, as in (Skalski and Gillian – 2001), we replace $(p - 1)$ with a continuous variable $y(y = y(t))$, since in our model, predator abundance is modeled as continuous variable. In conclusion, the literature review focused on various types of functional response and mathematical models with emphasis on two preys and one predator mathematical models. In this article, a mathematical model is formulated and analyzed to study the dynamics of one predator and two preys eco – system in which prey species are harvested and the first prey is easy to capture by the predator, while the alternative prey has adopted anti – predator behavior and so it requires a lot of scratching and handling time for the predator to capture.

3. MODEL FORMULATION AND ANALYSIS

Here, we are going to present model assumptions, formulation and analysis. Consider a prey – predator model in which Fox is predator species while the Antelope and Rabbit are the prey species.

The main feature of the model is that two different functional responses of the predator are incorporated in the model to represent the difference in the way the predator feeds on each of the prey species. The first prey is easy to capture prey and the predator's response to the easy to capture prey is Holling Type – II response. The second prey has adopted anti – predator behavior and is hard to capture prey and this behavior is represented by the Ratio – Dependent response of the predator. Constant effort harvesting of the prey is incorporated in the model to cater for the effects of human reaching on the prey species. Terms representing logistic grow of the prey species in absence of the predator are included in the model. That is why the model has three non – linear autonomous ordinary differential equations describing how the population densities of the three species would vary with time.

3.1. Assumptions

We are going to modify a mathematical model which describes the dynamics of the interaction between two prey populations and one predator population. So that, we take the following assumption to develop our analysis.

1. The species live in ecosystem where external factors such as droughts, fires, epidemics are stable or have similar effect on the interacting species.
2. One prey is easy to capture by the predator, while the other prey has adopted anti – predator behavior and so it requires a lot of searching and handling time for the predator to capture it.
3. The rate of human poaching of the prey is on average constant per unit tome and so it is represented as constant harvesting effort of the prey.
4. The prey population follows the logistic growth model in the absence of predator of human poaching.
5. The rate of increase of the predator depends on the amount of biomass it converts as food.

Let us take $X_1(t)$, $X_2(t)$, denote population densities of two prey, and $Y(t)$ denotes population density of the predator at the time t. Then, the mathematical model which describe the dynamics between the two preys and the one predator population is given by (V. Madhusudan and S. V. Jaya -2007) as

$$\begin{cases} \frac{dX_1}{dt} = S_1 X_1 \left(1 - \frac{X_1}{K_1}\right) - \omega_1 X_1 X_2 - \left(\frac{a_1 X_1}{1+b_1 X_1}\right) Y - H_1 X_2 \\ \frac{dX_2}{dt} = S_2 X_2 \left(1 - \frac{X_2}{K_2}\right) - \omega_2 X_1 X_2 - \left(\frac{c X_2}{1+d_1 X_2 + d_2 Y}\right) Y \\ \frac{dY}{dt} = -eY + \lambda_1 \left(\frac{a_1 X_1}{1+b_1 X_1}\right) Y + \lambda_2 \left(\frac{c X_2}{1+d_1 X_2 + d_2 Y}\right) Y \end{cases}$$
(3.1)

3.2. Parameters

The following are the parameters used in our model

1. S_1 and $S_2 \longrightarrow$ They are per capita intrinsic growth rates for prey X_1 and X_2 respectively.
2. K_1 and $K_2 \longrightarrow$ They are carrying capacities for prey X_1 and X_2 respectively.
3. ω_1 and $\omega_2 \longrightarrow$ They are coefficients for interspecific competition between prey X_1 and X_2 respectively.
4. a_1 and $c \longrightarrow$ They are capturing rates of predator Y on X_1 and X_2 respectively.
5. b_1/c and $d_1/c \longrightarrow$ They are the predator's handling time on prey X_1 and X_2 respectively.
6. H_1 and $H_2 \longrightarrow$ They are constant effort harvesting rate of prey X_1 and X_2 respectively.
7. $e \longrightarrow$ It is the mortality rate of predator Y.
8. $d_2 \longrightarrow$ measures the effect of anti – predator behavior of prey X_2, and
9. λ_1 and $\lambda_2 \longrightarrow$ They are coefficients which measure the predator's efficiency to convert prey biomass of X_1 and X_2 respectively into fertility (Reproductivity)

If we harvest the second prey by $H_2 X_2$ in the above model equation (3.1) is modified as follows:

$$\begin{cases} \frac{dX_1}{dt} = S_1 X_1 \left(1 - \frac{X_1}{K_1}\right) - \omega_1 X_1 X_2 - \left(\frac{a_1 X_1}{1+b_1 X_1}\right) Y - H_1 X_2 \\ \frac{dX_2}{dt} = S_2 X_2 \left(1 - \frac{X_2}{K_2}\right) - \omega_2 X_1 X_2 - \left(\frac{c X_2}{1+d_1 X_2 + d_2 Y}\right) Y - H_2 X_2 \\ \frac{dY}{dt} = -eY + \lambda_1 \left(\frac{a_1 X_1}{1+b_1 X_1}\right) Y + \lambda_2 \left(\frac{c X_2}{1+d_1 X_2 + d_2 Y}\right) Y \end{cases}$$
(3.2)

where H_2 It is constant effort harvesting rate of prey X_2

From our model assumptions and parameters, the equation which represents the dynamics of the one predator and two prey ecosystems are formulated below:

Without loss of generality, due to easy computations, non – depersonalization of the model represented by equation (3.2) is done so as to reduce the number of parameters as follows:

Let us take, $N_1 = b_1 X_1$; $N_2 = d_1 X_2$; $Z = d_2 Y$ and $\tau = S_1 t$

$$\begin{cases} \frac{dN_1}{dt} = N_1 \left(1 - \frac{N_1}{b_1 K_1}\right) - \frac{\omega_1 N_2}{S_1 d_1} - \left(\frac{a_1 Z}{S_1 d_1 + (1+N_1)}\right) - \frac{H_1}{S_1} \\ \frac{dN_2}{dt} = \frac{S_2}{S_1} N_2 \left[\left(1 - \frac{N}{d_1 K_2}\right) - \frac{\omega_2 N_1}{S_1 b_1} - \left(\frac{cZ}{d_2 S_1 (1+N_2+Z)}\right) - \frac{H_2}{S_1}\right] \\ \frac{dZ}{dt} = \frac{e}{S_1} Z \left[\left(\frac{\lambda_1 a_1 N_1}{e b_1 (1+N_1)}\right) + \left(\frac{\lambda_2 c N_2}{e d_1 (1+N_2+Z)}\right) - 1\right] \end{cases}$$
(3.3)

Again, let us take.

$\gamma_1 = \frac{1}{b_1 K_1}$; $\gamma_2 = \frac{1}{d_1 K_2}$; $\delta_1 = \frac{\omega_1}{S_1 d_1}$; $\delta_2 = \frac{\omega_2}{S_1 b_1}$; $g_1 = \frac{a_1}{S_1 d_2}$; $g_2 = \frac{c}{S_1 d_2}$;

$h_1 = \frac{\lambda_1 a_1}{e b_1}$; $h_2 = \frac{\lambda_2 c}{e d_1}$; $\alpha_1 = \frac{S_2}{S_1}$; $\alpha_2 = \frac{e}{S_1}$; $M = \frac{H_1}{S_1}$; $R = \frac{H_2}{S_1}$;

And then, the equation (3.3) becomes

$$\begin{cases} \frac{dN_1}{d\tau} = N_1 \left[1 - \gamma_1 N_1 - \delta_1 N_2 - \frac{Z g_1}{1+N_1}\right] - M \\ \frac{dN_2}{d\tau} = N_2 \left[\alpha_1 - \alpha_1 \gamma_2 N_2 - \delta_1 N_2 - \frac{Z g_2}{!+N_2+Z} - R\right] \\ \frac{dZ}{d\tau} = Z\alpha_2 \left[\left(\frac{h_1 N_1}{(1+N_1)}\right) + \left(\frac{h_2 N_2}{(1+N_2+Z)}\right) - 1\right] \end{cases}$$
(3.4)

All above parameters are in system 3.4 are positive only.

Remarks:

System (3.4) has 12 parameters compare to the system (3.2) has 16 parameters.

3.3. Existence of equilibrium points of the system

Now, we check the conditions of equilibrium points of the system (3.4). We can easily find that the system (3.4) has Seven Possible Non – Negative Equilibrium Points. They are

$$\begin{cases} E_0(0, \ 0, \ 0); \ E_1(N_1^*, \ 0, \ 0); \ E_2(0, \ N_2^*, \ 0); \ E_3(N_1^*, \ N_2^*, \ 0) \\ \qquad E_4(N_1^*, \ 0, \ Z^*); \ E_5(0, \ N_2^*, \ Z^*); \ E_6(N_1^*, \ N_2^*, \ Z^*) \end{cases}$$

Note that, the existence of $E_0(0, \ 0, \ 0)$ is trivial. Now, we will show the existence of other equilibrium points one by one.

1. **Existence of $E_1(N_1^*, \ 0, \ 0)$ *with* $N_1^* > 0$**

In this case, take, $N_2 = Z = 0$, and then in equation (3.4) we have,

$N_1 \left[1 - \gamma_1 N_1 - \delta_1 N_2 - \frac{Z g_1}{1+N_1}\right] \Rightarrow N_1^* = \frac{1-M}{\gamma_1} \Rightarrow E_1(N_1^*, \ 0, \ 0) = E_1 \left(\frac{1-M}{\gamma_1}, \ 0, \ 0\right)$

$\Rightarrow E_1$ Exists, if $M < 1$ \hfill (3.5)

From $(3.5) \Rightarrow h_1 < Z_1 \Rightarrow$ the absence of prey X_2 and predator Y, the harvesting rate of prey X_1 must be less than its intrinsic growth rate of equilibrium $E_1(N_1^*, \ 0, \ 0)$ to exist

2. **Existence of $E_2(0, \ N_2^*, \ 0)$ *with* $N_2^* > 0$**

In this case, take, $N_2 = Z = 0$, and then equation (3.4) gives,

$N_2[\alpha_1 - \alpha_1 \gamma_2 N_2 - R] \Rightarrow N_2^* = \frac{\alpha_1 - R}{\alpha_1 \gamma_2} \Rightarrow E_2(0, \ N_2^* 0, \ 0) = E_2 \left(0, \ \frac{\alpha_1 - R}{\alpha_1 \gamma_2}, \ 0\right)$, and hence the equilibrium point E_2 exists, if $R < \alpha_1$

3. Existence of $E_3(N_1^*, \quad N_2^*, \quad 0)$ with $N_1^* > 0$ and $N_2^* > 0$

In this case, take, $Z = 0$, and then in equation (3.4) we have,

$N_1[1 - \gamma_1 N_1 - \delta_1 N_2 - M] = 0$ and $N_2[\alpha_1 - \alpha_1 \gamma_2 N_2 - \delta_1 N_2 - R] = 0$

$\Rightarrow N_1^* = \left[\frac{\gamma_2 \alpha_1 - (1-M) + \delta_1(R - \alpha_1)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right]$ and $N_2^* = -\left[\frac{\gamma_1(R - \alpha_1) + \delta_2(1 - M)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right]$ \hfill (3.6)

$\Rightarrow E_3(N_1^*, \quad N_2^*, \quad 0) = E_3\left(\left[\frac{\gamma_2 \alpha_1 - (1-M) + \delta_1(R - \alpha_1)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right], \quad -\left[\frac{\gamma_1(R - \alpha_1) + \delta_2(1 - M)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right], \quad 0\right)$. And hence the equilibrium point E_3 exists, if

$M > 1; \ R < \alpha_1; \ \delta_2(1 - M) < \gamma_1(R - \alpha_1); \ \gamma_1 \gamma_2 \alpha_1 < \delta_1 \delta_2$ \hfill (3.7)

Note:

$E_3(N_1^*, \quad N_2^*, \quad 0) = E_3\left(\left[\frac{\gamma_2 \alpha_1 - (1-M) + \delta_1(R - \alpha_1)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right], \quad -\left[\frac{\gamma_1(R - \alpha_1) + \delta_2(1 - M)}{\gamma_1 \gamma_2 \alpha_1 - \delta_1 \delta_2}\right], \quad 0\right)$ it can also exists, if, $M > 1; \ R < \alpha_1; \ \delta_2(1 - M) < \gamma_1(R - \alpha_1); \ \gamma_1 \gamma_2 \alpha_1 > \delta_1 \delta_2$ \hfill (3.8)

Using Condition (3.7)$\Rightarrow \omega_1 \omega_2 < \frac{S_1 S_2}{K_1 K_2}$ and add $H_1 < S_1$ and $H_2 < S_2$, we get, in absence of the predator, the vital parameters for existence of the two prey species are; precipitate intrinsic growth rates of the prey, constant effort harvesting rate or the prey, carrying capacities of the prey and inter specific competition among the prey specie is negligible, then these two prey species will co – exist provided the constant effort harvesting rate is less than its precipitate intrinsic growth rate. We also comment that, because of the non – existence of inter specific competition among prey X_1 and X_2, so that, Condition (3.8) is not realistic as Condition (3.7)

4. Existence of $E_4(N_1^*, \quad 0, \quad Z^*)$ with $N_1^* > 0$ and $Z^* > 0$

In this case, take, $N_2 = 0$, and then equation (3.4) gives,

$N_1[1 - \gamma_1 N_1 - Z g_1 - M] = 0$ \hfill (3.9)

$Z\left(\frac{h_1 N_1}{1 + N_1} - 1\right)$ \hfill (3.10)

Using (3.9) and (3.10); for $Z \neq 0$ and $N_1 \neq 0$, we get

$N_1^* = \frac{1}{(h_1 - 1)}$ provided $h_1 > 1$ \hfill (3.11) and

$Z^* = \frac{1}{(h_1 - 1)g_1}[(h_1 - 1)(1 - M) - \gamma_1]$ \hfill (3.12)

$\Rightarrow E_4(N_1^*, \quad 0, \quad Z^*) = E_4\left(\frac{1}{(h_1 - 1)}, \quad 0, \quad \frac{1}{(h_1 - 1)g_1}[(h_1 - 1)(1 - M) - \gamma_1]\right)$, exists, if

$M < 1; \ h_1 > 1$ \hfill (3.13)

$(h_1 - 1)(1 - M) > \gamma_1$ \hfill (3.14)

Condition (3.13)$\Rightarrow \lambda_1 > \frac{eb_1}{a_1}$, that is, the λ_1 proportion of biomass of prey X_1 converted into food by the predator Y must be greater than the product of the predator's natural mortality rate, e and the time, it takes to handle the prey $\frac{b_1}{a_1}$

5. Existence of $E_5(0, \quad N_2^*, \quad Z^*)$ with $N_2^* > 0$ and $Z^* > 0$

In this case, take, $N_1 = 0$, and then equation (3.4) gives,

$N_2\left[\alpha_1 - \alpha_1 \gamma_2 N_2 - \frac{Z g_2}{1 + Z + N_2} - R\right] = 0$ \hfill (3.15)

$\Rightarrow Z\left(\frac{h_2 N_2}{1 + Z + N_2} - 1\right) = 0$ \hfill (3.16)

Equation (3.16) gives, for $Z \neq 0$ and $N_2 \neq 0$, we get

$Z = h_2 N_2 - N_2 - 1$ \hfill (3.17)

Substituting (3.17) in (3.15), we get

$h_2 \gamma_2 N_2^2 + (g_2 h_2 + R h_2 - h_2 - g_2)N_2 - g_2 = 0$ \hfill (3.18)

Equation (3.18), is the quadratic equation in terms of N_2, we have the form as

$A N_2^2 + B N_2 + C = 0$, where $A = h_2 \gamma_2$; $B = (g_2 h_2 + R h_2 - h_2 - g_2)$ and $C = g_2$

$\Rightarrow N_2^* = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ \hfill (3.19)

Since $A, B,$ and $C > 0$ then $N_2^* > 0 \Rightarrow$ From (3.17), $Z^* = (h_2 - 1)N_2 - 1 > 0$

$N_2^* = \frac{1}{(h_2 - 1)}$ provided $h_2 > 1$ \hfill (3.20)

The condition $h_2 > 1 \Rightarrow \lambda_2 > \frac{ed_1}{c}$, that is, λ_2 proportion of biomass of prey X_2 converted into fertility (ability to reproduce) by the predator Y must be greater than the product of the predator's natural mortality rate, e and the time, it takes to handle the prey $\frac{d_1}{c}$

6. Co – Existence Equilibrium Point $E_6(N_1^*, \quad N_2^*, \quad Z^*)$

Due to [Dubey and Upadhyay – 2000], equate equation (3.4) to zero and then we find two functionals, $f(N_1, N_2)$ and $g(N_1, N_2)$, which intersect at the equilibrium point N_1, N_2. Now, equating equation (3.4) to zero gives us

$$1 - \gamma_1 N_1 - \delta_1 N_2 - \frac{g_1 Z}{(1+N_1)} - M = 0 \tag{3.21}$$

$$\alpha_1 - \alpha_1 \gamma_2 N_2 - \delta_2 N_2 - \frac{g_2 Z}{(1+N_2+Z)} - R = 0 \tag{3.22}$$

$$-1 + \frac{h_1 N_1}{(1+N_1)} + \frac{h_2 N_2}{(1+N_2+Z)} = 0 \tag{3.23}$$

From equation (3.21), we get

$$Z = \frac{(1+N_1)}{g_1}(1 - M - \gamma_1 N_1 - \delta_1 N_2) \tag{3.24}$$

From equation (3.22), we get

$$Z = \frac{h_2 N_2 (1+N_1)(1-\gamma_2 N_2 - \delta_2 N_1 - R)}{g_2 - (1 - \alpha_1 \gamma_2 N_2 - \delta_2 N_2)} \tag{3.25}$$

Again equations (3.22) and (3.23) gives

$$Z = \frac{h_2 N_2 (1+N_1)(1-\gamma_2 N_2 - \delta_2 N_1 - R)}{g_2(1+N_1 - N_1 h_1)} \tag{3.26}$$

Equations (3.25) and (3.26) give

$$f(N_1, N_2) = \frac{h_2 N_2 (1+N_1)}{g_2(1+N_1-N_1 h_1)} - \frac{(1+N_2)}{g_2 - (1-\alpha_1 \gamma_2 N_2 - \delta_2 N_2)} = 0 \tag{3.27}$$

Equations (3.24) and (3.26) give

$$g(N_1, N_2) = \frac{(1-\gamma_1 N_1 - \delta_1 N_2 - M)}{g_1} - \frac{h_2 N_2 (1-\alpha_1 \gamma_2 N_2 - \delta_2 N_2 - R)}{g_2(1+N_1-N_1 h_1)} = 0 \tag{3.28}$$

Note that, equations (3.27) and (3.28) are two functions of N_1 and N_2. To prove the existence of $E_6(N_1^*, \quad N_2^*, \quad Z^*)$ under conditions those $f(N_1, N_2)$ and $g(N_1, N_2)$ meet in the interior of the positive (N_1, N_2) −plane, at a point (N_1^*, N_2^*) they can be easily found. Knowing $(N_1^*, N_2^*), Z^*$ they can be obtained from equation (3.25). From equation (3.27) as $N_1 \to 0$ and $N_2 \to N_{2f}$, that N_{2f} is

$$N_{2f} = \frac{-D_2 + \sqrt{D_2^2 - 4D_1 D_3}}{2D_1} \tag{3.29}$$

Where $D_1 = h_2 \gamma_2$; $D_2 = (g_2 D_1 - g_2 - h_2)$; and $D_3 = -g_2$

Clearly, N_{2f} is positive and real since $D_3 < 0$. Again, note that N_{2f} is same as N_2^* of $E_5(0, \quad N_2^*, \quad Z^*)$. From equation (3.28), as $N_1 \to 0$ and $N_2 \to N_{2f}$, that N_{2f} is given by

$$N_{2g} = \frac{E_2 + \sqrt{E_2^2 - 4E_1 E_3}}{2E_1} \tag{3.30}$$

Where $E_1 = g_1 h_2 \gamma_2$; $E_2 = -(g_2 \delta_1 + g_1 h_2 + g_1 h_2 R)$; and $E_3 = g_2(1-M)$

Clearly, N_{2g} is positive and real if $M > 1$. So that we get the result as N_{2f} and N_2 are points at which the functions $f(N_1, N_2)$ and $g(N_1, N_2)$ would cut the N_2 −axis in the (N_1, N_2) −plane respectively. From (3.25), $\frac{dN_2}{dN_1} = -\left(\frac{\partial f}{\partial N_1} / \frac{\partial f}{\partial N_2}\right)$

$$\frac{\partial f}{\partial N_1} = \frac{h_1 h_2 N_2}{g_2(1+N_1-N_1 h_1)^2} + \frac{\delta_2(1+N_1)}{[g_2-(1-\gamma_2 N_2 - \delta_2 N_1)^2]}; \frac{\partial f}{\partial N_2} = \frac{h_2(1+N_1)}{g_2(1+N_1-N_1 h_1)} + \frac{(g_2-\gamma_2)(\delta_2 N_1-1)}{[g_2-(1-\gamma_2 N_2 - \delta_2 N_1)^2]}$$

Note: $\frac{dN_2}{dN_1} > 0$, if $\frac{\partial f}{\partial N_1} > 0$ and $\frac{\partial f}{\partial N_1} < 0$, it requires $h_1 > 1$; $g_2 > \gamma_2$; $N_1 > \max\left\{\frac{1}{\delta_2}, \frac{1}{h_1-1}\right\}$,

where, $\frac{\partial g}{\partial N_1} = \left\{\left[\frac{\gamma_1}{g_1}\right] + \left(\frac{h_2 N_2}{g_2}\right)\left[\frac{(1-\gamma_2 N_2)(h_1-1)-\delta_2}{(1+N_1-N_1 h_1)^2}\right]\right\}; \frac{\partial g}{\partial N_2} = \left[\frac{\delta_1}{g_1} + \frac{h_2(1-\delta_2 N_1 - 2\gamma_2 N_2)}{g_2(1+N_1-N_1 h_1)^2}\right]$

Note: $\frac{dN_2}{dN_1} < 0$, if $\frac{\partial g}{\partial N_1} < 0$ and $\frac{\partial g}{\partial N_1} < 0$, that is, it requires $h_1 > 1$;

$N_1 > \max\left\{\frac{1}{\delta_2}, \frac{1}{(h_1-1)}\right\}$ and $0 < N_2 < \frac{1}{\gamma_2}\left[1 - \frac{\delta_2}{(h_1-1)}\right]$, since for $f(N_1, N_2)$ and $g(N_1, N_2)$ will meet if $N_{2f} < N_{2g}$ and therefore, "State the existence of the positive equilibrium point $E_6(N_1^*, \quad N_2^*, \quad Z^*)$, using the theorem as stated as

Theorem – 3.3.1:

"The positive equilibrium point $E_6(N_1^*, \quad N_2^*, \quad Z^*)$ will exist if, for $h_1 > 1$ and then

$$g_2 > \gamma_2; N_1 > \max\left\{\frac{1}{\delta_2}, \frac{1}{(h_1-1)}\right\}; 0 < N_2 < \frac{1}{\gamma_2}\left[1 - \frac{\delta_2}{(h_1-1)}\right] \text{ and } N_{2f} < N_{2g} \tag{3.31}$$

where N_{2f}, N_{2g}, they are defined in equations (3.29) and (3.30) respectively."

In terms of original parameters, $h_1 > 1 \Rightarrow \gamma_1 > \frac{eb_1}{a_1}, i.e. \gamma_1$ the proportion of biomass of preyX_1 converted into food by predator Y must greater than the product of the predator's natural mortality rate e and the time it takes to handle the prey $\frac{b_1}{a_1}$, and for the condition $g_2 > \gamma_2$ gives $c > \frac{S_2 d_2}{K_2 d_1}$

That is, the rate at which the predator capture prey X_2 should be greater than the product of the intrinsic growth rate of X_2 and the effect the anti – predator of preyX_2

3.4. Analyzing – Local Asymptotic – Stability of the Equilibrium Points (LASEPs)

(LASEPs), which is studied by computing the Jacobian Matrix (JM) and find the eigenvalues evaluated at each six equilibrium points. For stability of the equilibrium points, the real part of the eigenvalues o the (JM) must be negative

From equation (3.4) the (JM) of the system is given by

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial N_1} & \frac{\partial f_1}{\partial N_2} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial N_1} & \frac{\partial f_2}{\partial N_2} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial N_1} & \frac{\partial f_3}{\partial N_2} & \frac{\partial f_3}{\partial Z} \end{bmatrix} \text{ it gives } J(E_i) = \begin{bmatrix} P^{**} & -\delta_1 N_1 & -\frac{g_1 N_1}{(1+N_2)} \\ -\delta_2 N_2 & Q^{**} & -\frac{g_2 N_2(1+N_2)}{(1+N_2+Z)} \\ \frac{h_1 \alpha_2 Z}{(1+N_2)^2} - M & \frac{h_2 \alpha_2 Z(1+Z)}{(1+N_2+Z)^2} & O^{**} \end{bmatrix} \text{ (3.32)}$$

$Where\ P^{**} = 1 - 2\gamma_1 N_1 - \delta_2 N_2 - \frac{g_1 Z}{(1+N_1)^2} - M; \ Q^{**} = \alpha_1 - 2\gamma_2 \alpha_1 N_2 - \delta_2 N_1 - \frac{g_2 Z(1+Z)}{(1+N_2+Z)^2} - R$ And

$O^{**} - \alpha_2 + \frac{b_1 \alpha_2 N_1}{1+N_1} + \frac{h_2 \alpha_2 N_2(1+N_2)}{(1+N_2+Z)^2}$, now we start to analyze **(LASEPs)**for each:

1. $E_0(0, \ 0, \ 0)$; By using

Lemma – 1: "The boundary equilibrium pointE_0 of the system (3.4) is stable fixed point when $M > 1\ and\ R > \alpha_1$, otherwise unstable fixed point"

Proof:

By linearizing system (3.4) at E_0 we obtain (JM)

$$J(E_0) = \begin{bmatrix} 1 - M & 0 & 0 \\ 0 & \alpha_1 - R & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} \tag{3.33}$$

The eigenvalues of $J(E_0)$ are $1 - M; \ \alpha_1 - R\ and\ -\alpha_2$, if this condition hold when

$M > 1\ and\ R > \alpha_1$ (3.34)

Since all eigenvalues of $J(E_0)$ are negatives so that, E_0 is stable. Also, the equilibrium point with $M < 1\ and\ R < \alpha_1$ is always positive, and so that E_0 is unstable fixed point.

2. $E_1(N_1^*, \ 0, \ 0) = \left(\frac{1-M}{\gamma_1}, \ 0, \ 0\right)$; By using

Lemma – 2: "The boundary equilibrium pointE_1 of the system (3.4) is stable fixed point when $M < 1\ and\ R > \alpha_1; \gamma_1 < \delta_2(1 - M)\ and\ 0 < h_1 < 1$, otherwise unstable fixed point"

Proof:

By linearizing system (3.4) at E_1 we obtain (JM)

$$J(E_1) = \begin{bmatrix} 1 - M & \frac{\delta_{1(1-M)}}{\gamma_1} & \frac{g_{1(1-M)}}{\gamma_1 - (1-M)} \\ 0 & \left[\alpha_1 + \frac{\delta_{2(1-M)}}{\gamma_1}\right] - R & 0 \\ 0 & 0 & \alpha_2 \left[\frac{(1-M)(1-h_1) - \gamma_1}{\gamma_1 - (1-M)}\right] \end{bmatrix} \tag{3.35}$$

The eigenvalues of $J(E_1)$ are $(1 - M); \left[\alpha_1 + \frac{\delta_{2(1-M)}}{\gamma_1}\right] and\ \alpha_2 \left[\frac{(1-M)(1-h_1) - \gamma_1}{\gamma_1 - (1-M)}\right]$, the eigenvalues of these three are negatives, for$M < 1; \ R > \alpha_1; \gamma_1 < \delta_2(1 - M)\ and\ 0 < h_1 < 1$. (3.36)

$\Rightarrow$The equilibrium point$E_1(N_1^*, \ 0, \ 0)$ is locally asymptotically stable and for$M > 1; R > \alpha_1$ unstable. And also, if $M < 1 \Rightarrow H_1 < S_1; R > \alpha_1 \Rightarrow H_2 < S_2\ amd\ h_1 < 1 \Rightarrow \lambda_1 < c\left(\frac{b_1}{a_1}\right)$

$\Rightarrow$For local asymptotically stability of $E_1(N_1^*, \ 0, \ 0), H_1$, the harvesting rate of prey X_1,must be less thanS_1which is the intrinsic growth rate of prey $X_1\ and\ H_2$, the harvesting rate of preyX_2, must be greater than S_2, which is the intrinsic growth rate of prey $X_2$and also, λ_1, which measures the efficiency of the predator to convert the biomass of preyX_1, into fertility or reproductivity must be less than the predator's natural mortality rate e and the time, it takes to handle prey $X_1, \left(\frac{b_1}{a_1}\right)$, respectively.

3. $E_2(0, \quad N_2^*, \quad 0) = \left(0, \quad \frac{\alpha_1 - R}{\alpha_1 \gamma_2}, \quad 0\right)$; By using

Lemma – 3: "The boundary equilibrium point E_2 of the system (3.4) is locally asymptotically stable fixed point when $M > 1$; $\gamma_2 < R > \alpha_1$; $\gamma_2 < \delta_1(\alpha_1 - R)$; $R < \alpha_1$ and $0 < h_2 < 1$, otherwise unstable"

Proof:

By linearizing system (3.4) at E_2 we obtain (JM)

$$J(E_2) = \begin{bmatrix} \left[(1-M) + \frac{\delta_{1(\alpha_1-R)}}{\gamma_2}\right] & 0 & 0 \\ \frac{\delta_{1(\alpha_1-R)}}{\alpha_1 \gamma_2} & (\alpha_1 - R) & \frac{g_{2(\alpha_1-R)}}{\gamma_2 - (\alpha_1 - R)} \\ 0 & 0 & \left[\frac{e(\alpha_1-R)(1-h_2)-\gamma_1}{\gamma_2-(\alpha_1-R)}\right] \end{bmatrix} \qquad (3.37)$$

The eigenvalues of $J(E_2)$ are $\left[(1-M) + \frac{\delta_{1(\alpha_1-R)}}{\gamma_2}\right]$; $(\alpha_1 - R)$ and $\left[\frac{e(\alpha_1-R)(1-h_2)-\gamma_1}{\gamma_2-(\alpha_1-R)}\right]$, the eigenvalues of these three are negatives, for

$M < 1$; $R < \alpha_1$; $\gamma_2 < \delta_1(\alpha_1 - R)$ and $0 < h_1 < 1$. $\qquad (3.38)$

$\Rightarrow$The equilibrium point $E_2(0, \quad N_2^*, \quad 0)$ is locally asymptotically stable and for $M > 1$; $R > \alpha_1$ unstable. And also, if $M < 1 \Rightarrow H_1 < S_1$; that is, the constant effort harvesting rate of prey X_1 must be greater than its precipitate intrinsic grown rate and $R < \alpha_1 \Rightarrow H_2 < S_2$; that is, the constant effort harvesting rate of prey X_2 must be less than its percapita intrinsic grown of prey X_2 and using (3.38)

$\Rightarrow \lambda_2 < \frac{ed_1}{c}$, also, λ_2 measures the efficiency of the predator to convert to the biomass of prey X_2 into fertility or reproductively

must be less than the predator's natural mortality rate e and the time, it takes to handle prey X_2, $\left(\frac{d_1}{c}\right)$, for the local asymptotic stability

of E_2 $(0, \quad N_2^*, \quad 0)$

3. $E_3(N_1^*, \quad N_2^*, \quad 0) = E_3\left(\frac{(\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)})}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2}, \quad \frac{[\gamma_1(R-\alpha_1)-\delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1-\delta_1\delta_2}, \quad 0\right)$

The (JM→ $Jacobean$ $matrix$) evaluated at E_3 gives $J(E_3) = \begin{bmatrix} A^* & B^* & C^* \\ D^* & E^* & F^* \\ 0 & 0 & G^* \end{bmatrix}$ $where$

$A^* = -1 - 2\gamma_1 N_1^* - \delta_1 N_2^* - M$; $B^* = -\delta_1 N_2^*$; $C^* = g_1 N_1 f_1 + N_1^*$; $D^* = -\delta_2 N_2^*$;

$E^* = \alpha_1 - 2\gamma_2\alpha_1 N_2^* - \delta_2 N_2^*$; $F^* = \frac{g_2 N_2^*}{1} + N_2^*$; $G^* = \frac{h_2\alpha_2 N_1^*}{!+N_1^*} + \frac{h_2\alpha_2 N_1^*}{!+N_2^*}$

The eigenvalues of $J(E_3) = \begin{bmatrix} A^* - \lambda & B^* & C^* \\ D^* & E^* - \lambda & F^* \\ 0 & 0 & G^* - \lambda \end{bmatrix} = 0$ $\qquad (3.39)$

$\Rightarrow (A^* - \lambda(E^* - \lambda))(G^* - \lambda) - D^*B^* = 0$

$\Rightarrow \lambda^3 - \lambda^2(A^* + E^* + G^*) + (A^*G^* + E^*G^* + A^*E^* - D^*B^*)\lambda + G^*D^*B^* - G^*A^*E^* = 0$

$\Rightarrow \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

By Routh – Hurwitz criteria (Murray – 1989), $\lambda < 0$ if $a_1, a_2 > 0$. $a_1 a_2 - a_3 > 0$ $\qquad (3.40)$

Now, we discussed about each of these conditions as following manner:

For $a_1 = 0 \Rightarrow (A^* + E^* + G^*) > 0$ or $(A^* + E^* + G^*) < 0$

Now, we discussed about each of these conditions as following manner

Type – 1: $(\boldsymbol{For}\ \boldsymbol{G^*} < 0)$

$\Rightarrow -\alpha_2 + \frac{h_2\alpha_2 N_1^*}{!+N_1^*} + \frac{h_2\alpha_2 N_1^*}{!+N_2^*} < 0$, $where$ $N_1^* = \frac{(\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)})}{\gamma_1\gamma_2\alpha_1-\delta_1\delta_2}$ and $N_2^* = \frac{[\gamma_1(R-\alpha_1)-\delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1-\delta_1\delta_2}$, after simplifying we get,

$G^* = N_1^* N_2^*(h_1 + h_2 - 1) + N_1^*(h_1 - 1) + N_1^*(h_2 - 1) - 1 < 0$, it will hold when

$\begin{cases} M < 0; 0 < R < \alpha_1; \gamma_2(1-M) < \delta_1(\alpha_1 - R); \gamma_1(\alpha_1 - R) < \delta_2(\alpha_1 - M); \\ \quad\quad and \ \gamma_1\gamma_2\alpha_1 < \delta_1\delta_2; h_1 < 1; h_2 < 1 and \ h_1 + h_2 < 1 \end{cases}$ $\qquad (3.41)$

In terms of original parameters, $h_1 + h_2 < 1$ gives $\left(\frac{\lambda_1 a_1}{eb_1}\right) + \left(\frac{\lambda_2 c}{ed_1}\right) < 1$

$\Rightarrow$The predator's efficiency in converting the biomass of both prey into fertility or reproductively must be less than the predator's mortality rate and the time it takes to handle both preys.

Type – 2: $(\boldsymbol{For}\ \boldsymbol{A^*} < 0)$

$\Rightarrow 1 - 2\gamma_1 N_1^* - \delta_1 N_2^* - M < 0$ substituting N_1^* and N_2^* in this inequality, it gives

$A^* = 1 - 2\gamma_1 \left[\frac{(\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)})}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right] - \delta_1 \left[\frac{[\gamma_1(R-\alpha_1) - \delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right] - M < 0$

$\Rightarrow A^* < 0$, if $R < \alpha_1; M < 1; \gamma_1\delta_1(\alpha_1 - R) < \gamma_1\gamma_2(1 - M)$ and $\gamma_1\gamma_2\alpha_1 > \delta_1\delta_2$

Type – 3: ($\boldsymbol{For\ E^* < 0}$)
$\Rightarrow \alpha_1 - 2\gamma_2\alpha_1 N_2^* - \delta_2 N_1^* - R < 0$, substituting N_1^* and N_2^* in this inequality, it gives

$E^* = \alpha_1 - 2\gamma_2\alpha_1 \left[\frac{[\gamma_1(R-\alpha_1) - \delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right] - \delta_2 \left[\frac{(\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)})}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right] - R < 0$

$\Rightarrow E^* = \left[\frac{[\gamma_2\delta_2(1-M) - \gamma_1\gamma_2(R-\alpha_1)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right] < 0$

$\Rightarrow E^* < 0$, if $R < \alpha_1; M < 1; \gamma_2\delta_2(1 - M) < \gamma_1\gamma_2(R - \alpha_1)$ and $\gamma_1\gamma_2\alpha_1 > \delta_1\delta_2$ holds

Type – 4: ($\boldsymbol{For\ a_3 > 0}$)
$G^*(D^*B^* - A^*E^*) > 0$, it is satisfied for Type – I
$\Rightarrow (D^*B^* - A^*E^*) < 0$ (3.42)
$(D^*B^* - A^*E^*) = \alpha_1[N_1^*\mu_1 + N_2^*\mu_2 + (R - N_1^*\alpha_1 + M) - 4\gamma_1\gamma_2 N_1^* N_2^*]$ where
$\mu_1 = \delta_2 + 2\gamma_1 - 2\gamma_1 R - M\delta_2 - 2\gamma_1\delta_2 N_1^*$ and $\mu_2 = \delta_1 + 2\gamma_2 - \delta_1 R - M\gamma_2 - 2\gamma_1\delta_2 N_2^*$
For $(D^*B^* - A^*E^*) < 0$, if $\mu_1, \mu_2 < 0$ and $M + R < \alpha_1$ and substitute for

$N_1^* = \frac{(\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)})}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2}$ In μ_1 gives

$\mu_1 = \delta_2 + 2\gamma_1 - 2\gamma_1 R - M\delta_2 - 2\gamma_1\delta_2 \left[\frac{\gamma_2 - \alpha_{1[(1-m)+\delta_1](R-\alpha_1)}}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right]$ After simplification we get

$\mu_1 = \frac{[2\gamma_1^2\alpha_1\gamma_2(R-\alpha_1) - (\gamma_1\gamma_2 + \delta_1\delta_2)\delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2}$

$\Rightarrow \mu_1 = 0$; for $M < 1; R < \alpha_1; 2\gamma_1^2\alpha_1\gamma_2(R - \alpha_1) > (\gamma_1\gamma_2 + \delta_1\delta_2)\delta_1(1 - M)$ and $\gamma_1\gamma_2\alpha_1 < \delta_1\delta_2$

$\Rightarrow \frac{2\gamma_1^2}{\gamma_1\gamma_2 + \delta_1\delta_2} < \frac{\delta_2(1-M)}{\gamma_2(R-\alpha_1)}$ (3.43)

Similarly, substituting for $N_2^* = \frac{[\gamma_1(R-\alpha_1) - \delta_1(1-M)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2}$; in μ_2 gives

$\mu_2 = 2\gamma_2 + \delta_1 - \delta_1 R - 2M\gamma_2 - M\delta_2 - 2\gamma_2\delta_1 \left[\frac{\gamma_1(R-\alpha_1) + \delta_2(1-M)}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2} \right]$ After simplification we get

$\mu_2 = \frac{[2\gamma_1^2\gamma_1\alpha_1\gamma_2(1-M) - (\gamma_1\gamma_2 + \delta_1\delta_2)\delta_1(R-\alpha_1)]}{\gamma_1\gamma_2\alpha_1 - \delta_1\delta_2}$, and then $\mu_2 < 0$ if $M < 1; R < \alpha_1$ and $\gamma_1\gamma_2\alpha_1 < \delta_1\delta_2$

$\Rightarrow \frac{2\gamma_1^2}{\gamma_1\gamma_2 + \delta_1\delta_2} > \frac{\delta_1(R-\alpha_1)}{\gamma_1(1-M)}$ (3.44)

$\Rightarrow (D^*B^* - A^*E^*) < 0$ and if
$M < 1; R < \alpha_1; \gamma_2(R - \alpha_1) > \delta_2(1 - M); \gamma_1(1 - M) > \delta_1(R - \alpha_1)$
$\frac{2\gamma_1^2}{\gamma_1\gamma_2 + \delta_1\delta_2} = \max \left\{ \frac{\delta_1(R-\alpha_1)}{\gamma_1(1-M)}, \frac{\delta_2(R-\alpha_1)}{\gamma_2(1-M)} \right\}$ (3.45)

4. $\boldsymbol{E_4(N_1^*, \quad 0, \quad Z^*) = \left(\frac{1}{h_1 - 1}, \quad 0, \quad \frac{h_1}{(h_1-1)^2 g_1}[(h_1 - 1)(1 - M) - \gamma_1] \right)}$

Lemma – 4
The boundary condition of equilibrium point E_4 of the system (3.4) is locally asymptotically stable and it satisfies the conditions $M < \{1\ and\ h_1\}$
Proof
The Jacobean Matrix evaluate at E_4 and it gives

$J(E_4) = \begin{bmatrix} A_2^* & -\delta_1 N_1^* & -\frac{g_1 N_1^*}{1 + N_1^*} \\ 0 & B_2^* & 0 \\ \frac{h_1\alpha_2 Z^*}{(1 + N_1^*)^2} & \frac{h_1\alpha_2 Z^*}{(1 + Z^*)^2} & 0 \end{bmatrix}$ (3.46)

where $A_2^* = 1 - 2\gamma_1 N_1^* - \frac{g_1 Z^*}{(1 + N_1^*)^2} - M; B_2^* = \alpha_1 - \alpha_1\delta_2 N_1^* - \frac{g_1 Z^*}{(1 + Z^*)^2} - R$
The eigenvalues of $J(E_4)$ have negative real parts

If $A_2^*, B_2^* < 0$ and if $A_2^* < 0 \Rightarrow 1 - 2\gamma_1 N_1^* - \frac{g_1 Z^*}{(1+N_1^*)^2} - M < 0$ gives

$\frac{g_1 Z^*}{(1+Z^*)^2}[(1+N_1^*)^2(1 - 2\gamma_1 N_1^* - M) - g_1 N_1^*] < 0$ \hfill (3.47)

$\Rightarrow A_2^*, < 0, if \ (1 - 2\gamma_1 N_1^* - M),$ subsuming $N_1^* = \frac{1}{h_1 - 1}$ in eqn. (3.47), and simplify, we get

$(1 - M)(h_1 - 1) < 2\gamma_1$ and $B_2^* < 0$ and if $\alpha_1 - \delta_2 N_1^* - R < 0$ and substituting $N_1^* = \frac{1}{(h_1 - 1)}$, get

$(R - \alpha_1)(h_1 - 1) < 2\gamma_1$ \hfill (3.48) and

$0 < (R - \alpha_1)(h_1 - 1) < \delta_2$ \hfill (3.49)

5. $E_5(0, \quad N_2^* \quad Z^*)$

The Jacobean Matrix evaluate at E_5 and it gives

$$J(E_5) = \begin{bmatrix} A_3^* & 0 & 0 \\ -\delta_2 N_2^* & B_3^* & \frac{g_2 N_2^*(1+N_2^*)}{(h_2)^2 N_2^*} \\ h_2 \alpha_2 Z^* & \frac{h_2 \alpha_2 Z^*}{(1+Z^*)^2} & C_3^* \end{bmatrix}$$ \hfill (3.50)

$A_3^* = 1 - g_1 Z^* - M; B_3^* = \alpha_1 - 2\alpha_1 \gamma_2 N_2^* - \frac{g_2 Z^{/*}(1+Z^*)}{(h_2)^2 N_2^*}$ and $C_3^* = -\alpha_2 + \frac{\alpha_2(1+N_2^*)}{(h_2)^2 N_2^*}$

The eigenvalues of $J(E_4)$ have negative real parts

If A_2^*, B_2^* and $C_2^* < 0$ and if $A_2^* < 0$, these conditions will hold if

If $M < 0, R > \alpha_1$ and $h_1 > 1$ And then we have

$N_2^* < \min\left\{\frac{1}{\gamma_2}, \frac{1}{(h_1-1)}\right\}$ \hfill (3.51)

3.5.1. Global Stability of the Steady States

3.5.2. Global Stability of the Co -Existence Equilibrium point – E_6 (N_1^*, N_2^*, Z^*)

In this section, First, we choose a suitable Lapunove Function, from which conditions for the global asymptotic stability of the co – existence point E_6 (N_1^*, N_2^*, Z^*) are derived. For this case we prove a lemma, based on the work of (Takeuchi – 1996); (Chaudri and Kar -2002) and (Dubey and Upadhyay – 2004). And one theorem by our self

Lemma – 3.5. 1.

The set $\varphi = \left\{0 \le N_1 \le \frac{1}{\gamma_1}; 0 \le N_2 \le \frac{1}{\gamma_2}; 0 \le \sigma_1 N_1 + \sigma_2 N_2 + Z \le \frac{\rho}{\tau}\right\}$ where $\sigma_1 = \frac{e_1 h_1}{S_1 g_1}; \sigma_2 = \frac{e_2 h_2}{S_2 g_2};$

$\rho = \frac{\sigma_1}{\gamma_1}(S_1 + \tau) + \frac{\sigma_2}{\gamma_2}(S_1 + \tau)$ and $\tau < e$, it is a region of attraction for all solutions initiating in the interior of the positive region(N_1, N_2, Z^*)

Proof:

From the first equation of (3.4), note that $\frac{dN_1}{d\tau} < N_1(1 - \gamma_1 N_1)$

$\Rightarrow N_1 < \frac{\Gamma}{e^{-\pi\tau} + \gamma_1 \Gamma}$ where $\Gamma = \frac{N_1(0)}{N_1(0)N_2}$ as $\tau \to \infty$, and then we get

$N_1(\tau) = \frac{1}{\gamma_1}$ \hfill (3.53)

Similarly from the system (3.4) second equation we get as

$N_2(\tau) = \frac{1}{\gamma_2}$ \hfill (3.54)

Now, define a function $W(\tau) = \sigma_1 N_1(\tau) + \sigma_2 N_2(\tau) + Z(\tau)$ for real positive number η, and then we have

$\dot{W}(\tau) + \eta W(\tau) = \sigma_1 \dot{N}_1(\tau) + \sigma_2 \dot{N}_2(\tau) + \dot{Z}(\tau) + \eta\left(\sigma_1 N_1(\tau) + \sigma_2 N_2(\tau) + \dot{Z}(\tau)\right) = e$ \hfill (3.55)

Substituting for $\dot{N}_1(\tau), \dot{N}_2(\tau)$ and $\dot{Z}(\tau)$, using (3.4) into equation (3.53) and simplifying, we get

$\dot{W}(\tau) + \eta W(\tau) = \left\{ \begin{array}{l} \sigma_1 \dot{N}_1(S_1 + \eta) + \sigma_2 \dot{N}_2(S_2 + \eta) - \sigma_1 \gamma_1 N_1^2 - \sigma_2 \gamma_2 N_2^2 \\ -\sigma_1 S_1 \delta_1 N_1 N_2 - \sigma_1 S_1 \delta_2 N_1 N_2 - \sigma_1 S_1 M N_1 + (\eta - e)Z \end{array} \right\}$

Suppose that, if we choose $\eta < e$, we will get

$W(\tau) \le \frac{\sigma}{\eta}(1 - e^{-\eta\tau}) + \dot{W}(0)e^{-\eta\tau}$ as $\tau \to \infty \Rightarrow 0 \le W(\tau) \le \frac{e}{\eta}$

Hence completes the proof of Lemma

Theorem – 3.52

Let the following inequalities hold in the region φ as in (3.52), and then the co – existence equilibrium point E_6 (N_1^*, N_2^*, Z^*) is globally stable with respect to all solutions initiating in the interior of φ and then we have

$$\begin{cases} Z^* < \min\left\{\frac{\gamma_2 Q_2}{g_2}, \frac{\gamma_1 Q_1}{g_1}\right\}; \delta_1 + \delta_2 > 2\sqrt{\gamma_1 \gamma_2}; N_1^* = \frac{h_1 - g_1}{g_1} \\ N_2^* < \frac{h_2(g_1 + \gamma_1 Q_1) - g_1 g_2}{g_1 g_2}; \left(\gamma_1 - \frac{g_1 Z^*}{Q_1}\right) > \left(\frac{\delta_1 + \delta_2}{2}\right)^2; h_1 > g_1; h_2 > g_2 \end{cases}$$

Proof:

By using Lemma 3.51 and let us take the following Lapunove function

$$V(N_1, \quad N_2, \quad Z) = \frac{1}{S_1}\left[N_1 - N_1^* \quad - \quad N_1^* \ln\left(\frac{N_1}{N_1^*}\right)\right] + \frac{1}{S_2}\left[N_2 - N_2^* \quad - \quad N_2^* \ln\left(\frac{N_2}{N_2^*}\right)\right] + \frac{1}{e}\left[Z - Z^* \ln\left(\frac{Z}{Z^*}\right)\right]$$

Differentiating V with respect to time t; we get,

$$\dot{V}(N_1, \quad N_2, \quad Z) = \frac{(N_1 - N_1^*)}{S_1 N_1}\dot{N}_1^*(t) + \frac{(N_2 - N_2^*)}{S_2 N_2}\dot{N}_2^*(t) + \frac{(Z - Z^*)}{eZ}\dot{Z}^*(t)$$

Substituting in the above expression for N_1^*, N_2^*, and $\dot{Z}^*(t)$ from the system of equations (3.4), get

Hence completes the proof of theorem – 3.52.

4. NUMERICAL SIMULATION

In this section, we will do the numerical simulation of the model represented by equations (3.4) by using MATLAB software. (MATLAB – program is available in Appendix – A). Analytical studies become complete only with the numerical justification of the results. Qualitative analysis of the main features in the system is described by numerical simulations. Therefore, we assign some hypothetical data in order to verify the analytical result that has been obtained. The numerical experiments are conducted to examine the dynamical behaviour of the system. It is obvious that changing the parameter values change the numerical out comes. So, every different set of parameters give unique results.

4.1. The Effect of harvest preys $N1$ and $N1$

Harvesting of one or more animal species of an ecosystem can stabilize or destabilize the dynamics of the ecosystem. In a Linear Lotka – Volterra Model with constant effect harvesting of both preys and predators, it is known that constant effort harvesting raises the average number of preys per cycle and lowers the average number of predators. (Borelian and Coleman – 2004)
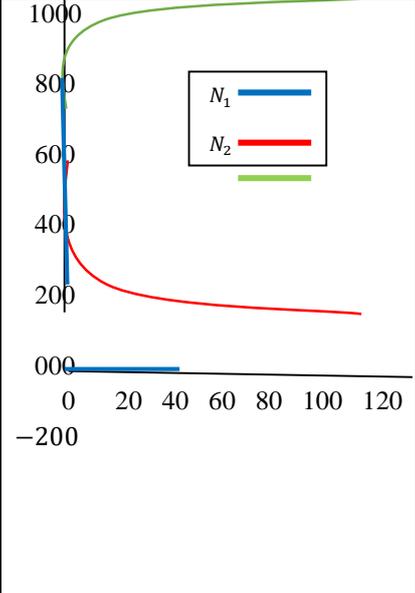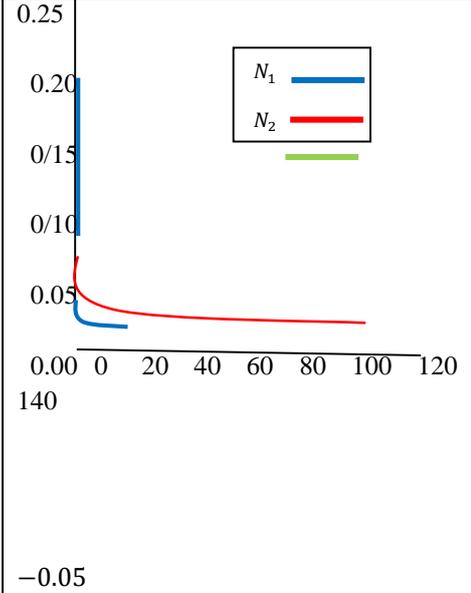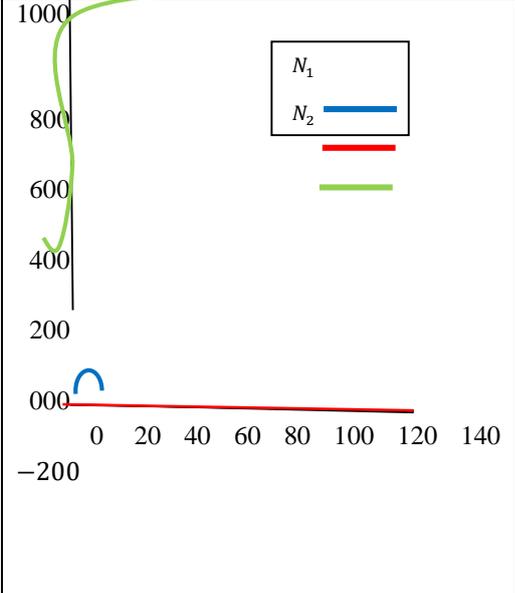
4.2. Select Parameters values from different authors assumptions

Table – 4.2: Parameters Values

Parameters	Values		Selected from Different articles
S_1 and S_2	1.820	and 0.730	Freedman – 1980, "Tropic Interactions of Prey – Predator System" Ecological Modelling, 35:1835, 1
γ_1 and γ_2	0.110	and 0.001	Freedman – 1980, "Tropic Interactions of Prey – Predator System" Ecological Modelling, 35:1835, 1
δ_1 and δ_2	1.000	and 0.001	Gleeson S. K. – 1994, "Density Dependency is better than ration Dependency" Ecology, 75: 1834 – 1835.
g_1 and g_2	0.040	and 0.001	Freedman – 1980, "Tropic Interactions of Prey – Predator System" Ecological Modelling, 35:1835, 1
M and R	3.000	and 2.000	Without loss of generality, Our Suitable Assumption
e	0.500		Gleeson S. K. – 1994, "Density Dependency is better than ration Dependency" Ecology, 75: 1834 – 1835
h_1 and h_2	0.125	and 0.500	Green, E. – 2004, "The effects of a mart predator in a one predator – two preys' system" (: http/green e/University of Chicago)

Assume that, the initial conditions N_1 and N_2, are 800 and 600 respectively, were used for the simulations.

Results – by Graphical Approach

Type – 1	Type – 2	Type – 3
Fig – 1: Graph of prey N_1, N_2 and Y against time when (M=0.01; R=0.02; $S_1 = 1.8$; and $S_2 = 0.73$) This graph shows that prey N_1 ↑ (increases when prey N_2 ↓ (decreases)highly as predator Y increases slightly	**Fig – 2: Graph of prey $N1$, $N2$ and Y against time when (M=0.02; R=0.05; $S_1 = 1.52$ and $S_2 = 0.73$) This graph shows that preys N_1 , N_2 and predator Y become extinct if the harvesting rate is greater or equal to its intrinsic growth rates S_1 and S_2 - Type – 2**	**Fig – 3: Graph of preys $N1$, $N2$ and Y against time when ($S_1 = 3 = M$; $S_2 = 2 = R$ This graph shows that prey N_1 , slightly increases while prey N_2 increases up to its carrying capacity and predator Y decrease to zero - Type –3**

5. DISCUSSION – CONCLUSION AND RECOMMENDATIONS

5.1. Discussion

A mathematical model was proposed and analyzed to study the dynamics of two – prey – one predator systems in which the predator showed a **Holling Type – II** response to one prey which was also harvested and a **Ratio – Dependent** Response to the other prey which was also harvested. All the **Seven Possible Equilibrium Points** were analyzed for **Local and Global Stability**. The harvesting rate was found to play a crucial role in **Stabilizing System**.

(Theorem – 3.3.1) showed that the three species would co – exist if N_1 and N_2 were not harvested beyond its intrinsic growth rate, the predator Y converted the biomass of prey N_1, into fertility at a rate greater than the predator natural mortality rate and the time it took to handle prey N_1. Also, the rate at which the predator captures prey N_2 should be greater than the product of the intrinsic growth rate of N_2 (Dubey and Upadhyay – 2004) and also discovered that the predator's mortality and food conversion coefficient played a crucial role in determining the stability behaviour of Planar Equilibrium in Ratio – Dependent Models. In (Akcakaya et al. – 1995), it was discovered that ratio – dependent models can have stable equilibrium, limit cycles and extinction of both preys.

5.2. Conclusion

In this article, conditions or existence of all the equilibrium points (Steady State were established/\

1. It was found that the prey N_1 they can exist on their own or in presence of the prey N_2 and/or the predator Y only if the intrinsic rate of the preys N_1 and N_2 were greater than the rate at which they are harvested.

2. The prey N_1 and prey N_2 would co – exist in the absence of the predator Y so long as the intrinsic rate of the preys N_1 and N_2 were greater than the rate at which, they are harvested and the inter specific competition among the preys N_1 and prey N_2 was negligible.

3. The existence of the predator Y with the either the prey N_1 alone or the prey N_2 alone required that the proportion of biomass of each prey species converted into fertility (Reproductive Rate) the predator must be greater than the product of the predator's natural mortality rate, and the time it takes to handle the prey

4. The co – existence of all three species required among others $H_1 > S_1$; $H_2 > S_2$; This inequality shows the parameters that must be controlled for the co – existence of the three species also,

5. The conditions for the local and global asymptotic stability of the steady states were established,

6. The conditions for the local asymptotic stability of the steady state were in most cases found to be similar to those for the existence of the steady states.
7. The global existence of the co – existence steady state E_6, stated in form of theorem (3.5.1)
8. The model is rich in dynamical behaviour and establishes various conditions under which the prey can exist with or without predation.

5.3. Summary

In this article, we have seen that whenever $H_1 > S_1$; $H_2 > S_2$ the system is not ecologically feasible as it gives negatives. This implies that the harvesting rate of the prey $N_1 \ and \ N_2$ should never be allowed to exceed its intrinsic growth rate. We note that whenever $S_1 \ and \ S_2$ increased, the population density of Y increased while that of $N_1 \ and \ N_2$ decreased. This clearly illustrates that as the intrinsic growth rate of the prey increases, the population density of the predator species will increase as well.

5.4. Recommendations

Basing on the results of Qualitative Analysis of the model, we recommended for researchers and others that

a. The prey species should not be harvested at a rate higher than their intrinsic growth rate. However Optimal harvesting of the preys $N_1 \ and \ N_2$ at a rate much lower than their intrinsic growth rate is permissible, since this would not lead to collapse of the system in the long term.
b. The population density of the predators can be increased drastically by increasing the intrinsic growth rate of the prey species.
c. The population density of the predators depends mainly on the biomass of the prey N_1 than of prey N_2, hence any attempt to control the population density of predators should be based on controlling the population density of the prey N_1

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Appendix - A: Mat lap Code S:

1.a. Prey Predator Model

Function start %--%

 % starting script to the module' $N1$ $N2$ Z model'

%--- %

% Implements the basic $N1\ N2\ Z$ model, and plots simulation results

% User Section 1: Definition of model parameters

% These parameters are passed to the function that calculates the derivatives.

% Note: Do not change the name 'param'

 Param. $S1 = 1.82$; % Set the parameter $S1$ of the model

Param. $S_2 = 0.73$ % Set the parameter $S2$ of the model

Param. $G2 = 0.01$; % Set the parameter $G1$ of the model

Param. $G2 = 0.01$; % Set the parameter $G2$ of the model

Param. $C1 = 1.00$; % Set the parameter $C1$ of the model

Param. $C2 = 0.001$; % Set the parameter $G2$ of the model

Param. $J1 = 0.04$; % Set the parameter $J1$ of the model

Param. $J2 = 0.01$; % Set the parameter $J2$ of the model

Param. $M = 0.01$; % Set the parameter M of the model

Param. $R = 0.02$; % Set the parameter M of the model

Param. $M = 0.5$; % Set the parameter E of the model

Param. $H1 = 0.125$; % Set the parameter $H1$ of the model

Param. $H2 = 0.5$; % Set the parameter $H2$ of the model

% This us the title string for the plot window, model_ title='prey predator model'; %

User Section 2: Definition of initial conditions

% Initial conditions are the values of all variables at time zero,

% Note: Do not change the name 'initial'! Define the initial values in the same order

% *as the d*erivatives *i*nitial, $N1 = 0.2$;

% set the initial value of '$N1$' Initial, $N2 = 0.01$; % set the initial value of '$N2$' Initial, $Z = 0.04$;

% set the initial value of 'Z'

% User Section 3: Definition of the simulation system $end\_time = 100$

% User Section 4: Definition of the ODE system

% % $function\ deriv = odesystem\ (t, x, param)$ % Function to calculate derivatives of the $S1$ model % Input:

 % t:Time (not used in this example because there is no explicit time dependence)

 % x: Vector of the current values of all variables in the same order as we can define the initial

% *v*alues: $(N1, N2, Z)$ % *p*aram: Used to past parameter values.

% Output:

% *d*erive: Column vector of derivatives, must be the same order as the input vector x,

$N1 = x(1)$;

$N1 = x(2)$;

$N1 = x(3)$;

$dN1 = $param.$S1 * N1 * (1 - (param.\ G1 * N1)) - (param.\ C1 * N2) - (Z * param.J1)/\ [(1 * N1) - (param\ .\ M)]$;

$dN2 = $param.$S2 * N2 * (1 - (param.\ G2 * N2)) - (param.\ C2 * N1) - (Z * param.J2)/\ [(1 * N2 + Z) - (parm.\ R)]$; $dZ = $param.$E * Z * (1 - (param.\ H1 * N1))/(1 + N1)(param.\ H2 * N2)/(1 + N2 + Z)$;

$deriv = [d1; dN2; dZ]$;

end

%

% Now we solve the ODE system and plot the results

%

% Extract initial values from the 'initial' structure and collect them in a column vector for use in 'ode45' $initial\_values, =[\ ]$);
$variable\_names = fieldnames\ (initial\ );\ for\ i = 1: length\ (variable\_names)\ initial\_values = [initial\_values; initial,$
$(variable\_names\ (i))];$
end

% Integrate the ODE system
$[t, y] = ode45(C(t, x)ode\_system\ (t, x, param), \ldots)[0,$
$end\_time]\ \ldots\ .\ ,\ initial)\_values, \ldots,[\ ];$

% prepare legend texts
$legend)\_texts= cell(kength(variable\_names), 1)$
$for\ i = 1: length\ (variable\_names)$
$text =[\ variable\_names\ (i),\ (t)'];\ legend\ texts(i) = text;$
end
 % plot the results
$plot(t.\ y, ' inewidth ' , 3);$
$xlable('time ')$
$ylable('number\ preypredator')$