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Existence of Seismic Wave Due to Tall Building

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ABSTRACT

In this article, we proposed and analyzed a mathematical model for the so – called city – effect in what in which an earthquake can be locally disturbed by the collective response of tall buildings in the large city. We use a set of equations c0upling vibrations buildings in buildings to motion under the ground. These equations were previously studied by different authors the existence of the seismic wave was ensured by a method of Numerical solutions. None those authors do not consider differential methods to show the existence of earthquake due to tall buildings. Furthermore, we show how we can solve for the wavenumber in a non – linear equation and we are able to find resonant frequencies coupling seismic waves and vibrating tall building which ensure the existence seismic wave due to building.

KEYWORDS: Accelerographs, Seismographs

1. INTRODUCTION

1.1. Background of the Study

Seismic waves are the waves of energy caused by the sudden breaking of rock within the earth or seismic waves are the waves of energy produced by an earthquake [7], [8]. They are the energy that travel through the earth and recorded on seismograph. An earthquake is a trembling of Earth caused by sudden release of stored energy, usually along faults. It has been observed that when an earthquake strikes a large city, the seismic activity is altered by the collective response of the buildings of the city [14]. This activity is called "the City = Effect" [9] and these seismic waves primarily of two types called

a. The Body Waves and Surface Waves

These two types together cause shaking of ground (Surface of the Earth) on which the buildings are founded. The characteristics of the ground shaking control earthquake responses of buildings, in addition to the building characteristics. The ground motion can be measured in the form of acceleration, velocity or displacement. Earth scientists (Geologists) are interested in capturing the size and origin of the earthquakes worldwide and measure feeble ground displacements even at great distances from the epicenter of the earthquakes. Instruments that measure these low-level displacements are called **Seismographs**. The vicinity of the epicenters of large earthquakes the ground shaking is violent [2]. Seismographs get started as their designs is such that they get saturated under large displacement shaking and become ineffective in capturing the displacement of the ground. And on the other hand, engineers are interested in studying levels of ground making at which buildings are damaged and are conversant with forces (As part of the design process of buildings). Hence this motivated the developments of instruments called **Accelerographs** that recorded during the earthquake shaking acceleration as function of time of the location where the instrument is placed. These instruments successfully capture the ground shaking even in the near field of the earthquake faults, where the shaking is violent [4].

The two most important variables affecting earthquake changes are

1. The intensity of ground shaking caused by the quake coupled and the quality of the engineering of structures in the region
2. The level of shaking in turn, is controlled by the proximity of the earthquake source to the affected region and the types of rocks that seismic waves pass through in route (particularly these at the stronger the shaking.

But there have been large earthquakes with very little damage either because they caused little shaking or because the buildings more are, they caused little shaking or because the building were built to withstand that kind of shaking. In other words, moderates' earthquakes have caused significant damage either because the shaking was locally amplified or more likely because the structures were poorly engineered [11] During an earthquake, building oscillate. But not all buildings respond to an earthquake equally. Small buildings are more affected or shacked by high frequency waves (short and frequent). For example,

"A small boat sailing in the ocean will not be greatly affected by a large swell. On the other hand, several small waves in quick succession can overturn or capsize the boat",

In much the same ways small building experiences more shaking by high – frequency earthquake waves. Large structures or high – rise buildings are more affected by long period or slow shaking. For instance; "An ocean lines will experience little disturbance by short waves in quick succession". However, "A large swell will significantly affect the ship. Similarly, a skyscraper will sustain greater shaking by long – period earthquake waves than by the shorter waves [16].

2. DISCUSSION AND ANALYSIS

2,1. The Associated Spectral Problem

The time harmonic solution for the city – effect modeling system of equation is as follows. We set

$$\omega(t, \quad x, \quad y) = \Phi(x, \quad y)^{-i\mu t} \quad (2.1.1)$$

Here, $\Phi: \Omega \longrightarrow \mathbb{R}$ represents the soil displacement $\mu > 0$ is the rescale frequency. Now let us take

$\alpha = u_j;\ \eta = v_j$ of the buildings. Then after necessary differentiation in a system $(2.1)\ to\ (2.5)$ we obtain the corresponding eigenvalues. Now we assign

$$-s\Delta\ \Phi = \mathrm{p}\mu^2\Phi\ in\ \Omega \quad (2.1.2)$$

$$k(\eta-\alpha)\mu^2 m_1\eta;\ \mathbb{R}(\Phi) - k(\eta-\alpha) = \mu^2 m_o\alpha \quad (2.1.3)$$

$$\Phi = \alpha, on\ \Gamma_j\ \ and\ \frac{\partial\Phi}{\partial y} = 0\ \ on\ \Gamma_{Free} \quad (2.1.4)$$

The last step is the non – dimensionalization of the problem [(4.2) – (4.4)]. We introduce a characteristic length 1, The non – dimensional spatial coordinates and non – dimensional frequency are

$$x' = \frac{x}{l};\ y' = \frac{y}{l};\ \xi = \mu\frac{1}{\beta} \quad (2.1.5)$$

Note that, we will omit primes and write $x\ and\ y$ in future, but these are the non – dimensional coordinates. Set Ω; Γ; Γ_{free} changes accordingly, but we will keep the notations. The non – dimensional city parameters are:

$$\gamma_b = \frac{m_1}{m_0};\ f_b = \frac{l_b}{l};\ \Upsilon = \frac{\rho_b}{\rho};\ b = \frac{\beta_b}{\beta} \quad (2.1.6)$$

First, we set,

$$\rho(\xi^2) = c_b^2\xi^2 - b^2 f_b^2 = \frac{2c_b^2\xi^2}{f_b}\left(c_b^2\xi^2 - \frac{\gamma_b+1}{\gamma_b}\right)\rho(\xi^2) \quad (2.1.7)$$

We can calculate $\alpha_j\ andd\ \eta_j$ as

$$\alpha_j = \phi(x,0)\ for\ (x,0)\ in\ \Gamma_j\ \ and\ \eta_j = \frac{bx^2 f_b^2 c x_j}{\rho(\xi^2)} \quad (2.1.8)$$

After simple calculations we will get finally, the following non – linear eigenvalue problem

$$\Delta\phi + k^2\varphi = 0\ in\ \Omega \quad (2.1.9)$$

$$\frac{\partial\phi}{\partial y} = 0\ in\ \Gamma_{free} \quad (2.1.10)$$

$$q(k^2)\ \phi(x,0) = p(k^2)\int_{\Gamma_j}\frac{\partial\phi}{\partial y}(s,0)ds\ \ for\ (x,0)\ in\ \Gamma_j\ \ and\ j\ in\ [1,N] \quad (2.1.11)$$

For $k =\ \xi\ where\ k$ is the value of wavenumber and also here $k > 0$. Our main objective is found values of the non – dmensionalized frequency k for which system [(4.9) – (4.11)] will be solvable, For a careful derivation of how given the mass density, the shear rigidity of the building and the ground, the height and the width of the building, the mass at the top of the building and the mass of the foundation of the building, after non – dimensionalization, and then we arrive the following system of equations, and assume that the building has rescaled width is 1 and is standing on the x_1 −axis, so that, its foundation mat=y be assumed to be the line segment $\Gamma = \left[-\frac{1}{2},\frac{1}{2}\right]$, we get,

$$\Delta\phi + k^2\varphi = 0\ in\ \mathbb{R} \quad (2.1.12)$$

$$\phi = 1\ in\ \Gamma \quad (2.1.13)$$

$$\frac{\partial\phi}{\partial y} = 0\ in\ (y=0)/\Gamma \quad (2.1.14)$$

$$\frac{\partial\phi}{\partial r} = -ik\ \varphi = O(r^{-1}) \quad (2.1.15)$$

$$q(k^2) = p(k^2)\int_{\Gamma_j}\frac{\partial\phi}{\partial y}(s,0)ds \quad (2.1.16)$$

$$where, p(t) = c_1 t - c_2;\ \ q(t) = t(c_3 t - c_4) \quad (2.1.17)$$

Here, $k > 0$ is the wavenumber and $r = \sqrt{x_1^2 + x_2^2}$, the rescaled physical displacement is $\mathbb{R}e\phi e^{-kt}$ and the constants $c_1, c_2, c_3. c_4$ are determined by the physical properties of the underground and the building. Note that the system [(4.12) – (4.17)] will be non – linear in the unknown wavenumber k, The target of this article is to show the following theorem.

Theorem – 1:

For any values of the constants $c_1, c_2, c_3. c_4$ and the system [(4.12) – (4.17)] have at least one solution in k, that is, there exists a positive k and a function and it has locally H^1 regularity in $\mathbb{R}^2$ such that equations [(4.12) – (4.17)] are satisfied. Moreover, the system of equations has at most finite number of solutions in k. In this theorem, the standard arguments can show that if we have x positive k system of equations [(4.2) through (4.15) is uniquely solvable. This theorem asserts that for some of those $k's$ the additional relation (4.16) will hold.

Proof:

$$\text{Define } F(k) = q(k^2) = p(k^2)\mathbb{R}e \int_{\Gamma_j}^{\cdot} \frac{\partial \phi}{\partial y}(s,0)ds \text{ for } k \text{ in } (0,\infty) \tag{2.1.18}$$

First, we will show that F is a real analytic in k. Then we will perform a low frequency and a high frequency analysis of ϕ. The low frequency analysis of ϕ will show that Fmust be negative in $(0,\infty)$ for some positive number. The high frequency analysis will prove that $\lim_{k\to\infty} F(k) = \infty$, it is concluding the proof of this theorem.

3. THE BOUNDARY DIRICHLET CONDITIONS TO NEUMANN OPERATOR ANALYTICITY WITH RESPECT TO WAVENUMBER.

Consider, D it is the Open Unit Disk in $\mathbb{R}^2$ centered at the origin. First, we will discuss well known lemma, and it will help in the detailed study of the related Dirichlet to Neumann operator relevant to our study.

3.1. Lemma – 1:

Let $k > 0$ is the wavenumber and f is the function in the Sobolov space $H^{1/2}(\partial D)$ and then the problem is

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2 \backslash D_e \tag{3.1}$$

$$u = f\partial D \tag{3.2}$$

$$\frac{\partial u}{\partial r} - iku = O(r^{=1}) \text{ uniformly as } r \longrightarrow \infty \tag{3.3}$$

This system has unique solution written as $f = \sum_{n=-\infty}^{+\infty} a_n e^{-in\theta}$, we have

$$u = \sum_{n=-\infty}^{+\infty} a_n e^{-in\theta} \left[\frac{H_{-n}(kr)}{H_n(kr)}\right] \tag{3.4}$$

This series and all its derivatives are uniformly convergent on any subset on $\mathbb{R}^2$ $for\ r \geq A\ where\ A > 1.$

Proof:

When we are dealing with an interior problem, we seek of progressive wave solutions. Considering the fact, plane waves are solution to the impose additional boundary condition at infinity to equation o Dirichlet and Neumann problem in order to guarantee uniqueness.

To determine Plane wave it is sufficient to look for solution u that decrease at infinity as $\frac{1}{r}$ $when$ $r = \sqrt{x_1^2 + x_2^2}$ is the distance from the origin.

Yet, this is not enough to guarantee uniqueness for example $u = \frac{\sin(kr)}{r}$ is a non-zero solution to the Dirichlet and Neumann Problem in free space.

Thus, we add the extra Summerfield radiation condition or outgoing wave condition, that is

$$\left|\frac{\partial u}{\partial r} - iku\right| \leq \frac{c}{r^2} \tag{3.5}$$

The change of convention in equation (3.3) and it yields a radiation condition with the opposite sign

$$\left|\frac{\partial u}{\partial r} 6iku\right| \leq \frac{c}{r^2} \tag{3.6}$$

We expect that this ingoing Summerfield condition will as well lead to a well – posed problem which we are going to prove.

$$\Delta u + k^2 u = 0 \text{ in } \Omega \tag{3.7}$$

$$u|_r = u_d \tag{3.8}$$

$$\left|\frac{\partial u}{\partial r} - iku\right| \leq \frac{c}{r^2} \tag{3.9}$$

Where, u_d it is the Dirichlet Data

This system [(3.5) to (3.9)] has not unique solution in the sphere (Space). We have seen in the case of sphere. We need to add the radian condition. A form the condition is $\left|\frac{\partial u}{\partial r} - iku\right| \leq \frac{c}{r^2}$ A weaker for is $\int_{\Omega}^{\cdot} \left|\frac{\partial u}{\partial r} - iku\right| \leq c$ (3,10)

3,3, The0rem – 2 – Uniqueness Theorem

The exterior Dirichlet and Neumann Problem [(3.5) to (3.9)] admit at most one solution in the Hilbert Space H

Proof:

The difference between two solution $u_1\ and\ u_2$ has a zero boundary conditions. That is, satisfies

$$u|_r = 0 \ or \ \left(\frac{\partial u}{\partial r}\right)\Big|_r = 0$$

Multiply, the above equations by complex conjugate $\bar{u}$ and integrate by parts to obtain

$$\int_{\Omega_n B_R}^{\cdot} [] \nabla u[^2 - k^2|u|^2]\, dx - \int_{S_R}^{\cdot} \frac{\partial u}{\partial r}\bar{u}\, d\sigma = 0 \tag{3,11}$$

At the exterior of the ball B_R, we expand the solution in the spherical harmonics. It holds that

$$u(r,\theta,\varphi) = \sum_{L=0}^{\infty}\sum_{m=t}^{L}\left[\alpha_L^m\left(\frac{h_l^{(1)}(kr)}{h_l^{(1)}(kR)}\right) + \beta_1^{(1)}\alpha_L^m\left(\frac{h_l^{(2)}(kr)}{h_l^{(1)}(kR)}\right)Y_l^{(m)}\right](\theta,\varphi) \tag{3.12}$$

And $\left(\frac{\partial u}{\partial r} - iku\right)$ has the expansion

$$\left(\frac{\partial u}{\partial r} - iku\right) = \begin{Bmatrix} \frac{k\alpha_L^m}{h_l^{(1)}(kR)}\left(\frac{d}{dr}\left(h_l^{(1)}(kr) - ih_l^{(1)}(kr)\right)\right) + \\ \frac{k\beta_1^{(1)}\alpha_L^m}{h_l^{(1)}(kR)}\left(\frac{d}{dr}h_l^{(2)}(kr)\right)Y_l^{(m)} \end{Bmatrix}(\theta,\varphi) \quad (3.13)$$

Due to the orthogonality of the spherical harmonic implies that the integral on the complement of B_R of the form

$$u = \sum_{L=0}^{\infty}\sum_{m=t}^{L}\left[\theta_l^{(m)}(r)Y_l^{(m)}\right] for\ r > R \quad (3.14)$$

And it is,

$$\int_{B_R}^{\cdot}|u|^2 dx = \sum_{L=0}^{\infty}\sum_{m=t}^{L}\int_R^{\cdot}\left[\left|\theta_l^{(m)}(r)\right|^2 r^2\right]dr \quad (3.15)$$

The dominant term in the expression $\left(\frac{\partial u}{\partial r} - iku\right)$ is the one coming form $h_L^{(2)}$, It behaves as

$$-2i\left(\frac{e^{-ikr}}{r}\right)\left(\frac{\beta_L^m}{h^{(2)}(kR)}\right)$$

Thus, the convergence of the associated series is possible only when all the coefficients

$$\beta_L^m = 0 \quad (3.16)$$

We have proved that the solution only the terms $h_l^{(1)}$. The imaginary part of this expression reduces

$$\int_{S_R}^{\cdot}(T_R u, \overline{u}) = 0 \quad (3.17)$$

$Where$, T_R is the capacity operator on the ball with radius R. From the expression of the operator follows

$$\alpha_L^m = 0 \quad (3.18)$$

From which we infer that,

$$\frac{u}{S_R} = 0 \quad (3.19)$$

$$\frac{\frac{\partial u}{\partial r}}{S_R} = 0 \quad (3.20)$$

Thus, the solution van9sheson the exterior of the ball β_R. Moreover, the Helmholtz equation is the elliptic operator, from which it follows that the solution is analytic in the domain Ω_e, it is zero.

Remarks:

There exists a weaker form for the radiation condition which is

$$\lim_{R\to\infty}\int_{S_R}^{\cdot}\left|\frac{\partial u}{\partial r} - iku\right|^2 d\sigma = 0 \quad (3.21)$$

In the same way as above, we get $\beta_l^{(m)} = 0$ and so that we get uniqueness of solution.

Another weaker form for the radiation condition is used the following notes.

Note that, the Fredholm alternative, we know that uniqueness existence shows the uniformly convergence of the series

$$u = \lim_{n=-\infty} a_n e^{-in\theta}\frac{H_n(kr)}{H_n(k)}$$

We need to consider the usefulness of Hankel Function.

3.2. Lemma 3.3.

The following equivalence as $n \longrightarrow \infty$ is uniform for all z in a compact set of $(0, \infty)$

$$H_n(z) \sim -\frac{i}{\pi}\left(\frac{1}{2}z\right)^{-n}(n-1)i \quad (3.22)$$

Proof:

For any positive z

$$\left|J_n(z)ni\left(\frac{1}{2}z\right)^{-n} - 1\right| = \frac{1}{n+1}\left|\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)\frac{z^{2k}(n+1)i}{ki(n+k)i}\right| \le \frac{1}{n+1}\left(\frac{1}{4}\right)^k\frac{z^{2k}}{ki} \quad (3.23)$$

It is clear that $J_n \sim \left(\frac{1}{2}z\right)^n\left(\frac{1}{ni}\right)$ is uniformly, for all z, in a compact set of $(0, \infty)$ and then we conclude that it is true.

We have

$$H_n(k) \sim -\frac{i}{\pi}\left(\frac{1}{2}k\right)^{-n}(n-1)i\ and\ H_n(kr) \sim -\frac{1}{\pi}\left(\frac{1}{2}kr\right)^{-n}(n-1)i$$

$$\Rightarrow \left(\frac{H_n(kr)}{H_n(k)}\right) = \frac{1}{r^n}$$

$\Rightarrow \left(\frac{H_n(kr)}{H_n(k)}\right) = \frac{1}{r^n}$ uniformly in r as long as r remains in a compact set of $(0, \infty)$

Let $A\ and\ B$ are two real numbers such that $1 < A < B$ and a set $M = \sup|a_n|$

$$\Rightarrow \left|a_n e^{in\theta}\left(\frac{H_n(kr)}{H_n(k)}\right)\right| \le \frac{2M}{r^n} = \frac{2M}{A^n}$$

$\Rightarrow \left|a_n e^{in\theta}\left(\frac{H_n(kr)}{H_n(k)}\right)\right| \le \frac{2M}{A^n}$ it is bounded and closed.

$\Rightarrow$For al n as large as enough, uniformly for all r in $[A, B]$

$\Rightarrow$The above series is uniformly convergent on any compact set of $\mathbb{R}^2/\overline{D}$

3.3. Lemma – 3.4:

For $z > 0$ the following limit as the following limit as $z \longrightarrow 0$ is uniform for all integers n different from 0

$$\lim_{z\to 0^+} -\frac{z}{|n|}\frac{H_n'(z)}{H_n(z)} = 1 \quad (3,24)$$

For the special case $n = 0$, we have $\lim_{z\to 0^+} -\frac{z}{|n|}\frac{H_n'(z)}{H_n(z)} = 0$

Proof:

Observe for $n \ge 2$

$$\left[H_n(z)\left(-\frac{i}{\pi}\left(\frac{1}{2}z\right)^{-n}\left((n-1)i\right)^{=1} = 1\right)(n-1)\right]$$

It can be bounded by a function in z and it is continuous on $[0, \infty)$ and independent of n therefore

$$H_n(z) \sim -\frac{i}{\pi}\left(\frac{1}{2}k\right)^{-n}(n-1)u(3,25)$$

As $z \longrightarrow 0^+$

Uniformly on $(0,b)$ for fixed b using formula

$H_n'(z) = -H_{n+1}(z) + \frac{n}{z}H_n(z),$

(3.26)

we get

$\frac{z}{|n|}\frac{H_n'(z)}{H_n(z)} \sim 1 \; as \; z \longrightarrow 0^+$ uniformly on $[0,b]$ for fixed $b > 0$

For $n \leq -2$, apply the formula

$H_{-n}(z) = (-1)^n H_n(z)$

(3.27)

Next to show that, all its derivatives are uniformly convergent on any subset of f$\mathbb{R}^2$ in the form $r \geq A$ where $A > 1$

Thus, we get,

$$u_r = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} k\left(\frac{H_n'(kr)}{H_n(k)}\right)$$

And then we simply $k\left(\frac{H_n'(kr)}{H_n(k)}\right)$ to do this, let us use derivatives of Hankel function. That is,

$$H_n'(kr) = -H_{n+1}(z) + \frac{n}{z}H_n(z)$$

(3.28)

$$\Rightarrow k\left(\frac{H_n'(kr)}{H_n(k)}\right) \sim -k\left(\frac{H_{n+1}'(kr)}{H_n(k)}\right)\frac{n}{r}k\left(\frac{H_n(kr)}{H_n(k)}\right)$$

Thus, the system $-k\left(\frac{H_n+1(kr)}{H_n(k)}\right) \sim -\frac{2n}{r^{n+1}}$ and the term $\frac{n}{r}k\left(\frac{H_n(kr)}{H_n(k)}\right) \sim \frac{n}{r^{n+1}}$, and then we have

$$\left(\frac{H_n+1(kr)}{H_n(k)}\right) \sim -\frac{n}{r^{n+1}}$$

Set $M = \sup|a_n|$, it follows that

$$\left|a_n e^{in\theta} k\left(\frac{H_{n+1}(kr)}{H_n(k)}\right)\right| \leq \frac{2M}{A^{n+1}} \; for \; e^{in\theta} \leq 2$$

$\Rightarrow$At this stage we can conclude that the r derivatives of the given series

$u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)$ is uniformly convergent on any compact set of f$\mathbb{R}^2$

Now, we show that the convergence of a θ derivatives of the serries

$$u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)$$

That is, u_θ is a multiplicative of a series $u \; by \; in \; 'in'$ that is

$u_\theta = \sum_{n=-\infty}^{\infty} ine^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)$ and

$$\left| u_\theta = \sum_{n=-\infty}^{\infty} ine^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)\right| \leq \frac{2Min}{A^n} \; when \; M = \sup|a_n| \; and \; \left(\frac{H_n(kr)}{H_n(k)}\right) \sim \frac{1}{r^n}$$

That is, u_θ is closed and bounded series that implies that u_θ is uniformly converges.

3.4. Lemma – 3.5:

For any $n \; in \; N, |H_n(z)|$ is a decreasing function of z on $(0,\infty)$

Case – 1;

For $n \leq -2$ it is obviously true

Case – 2:

For $n > 2$ we conclude that from this lemma, and $H_n(kr) \sim \frac{i}{\pi}\left(\frac{1}{2}z\right)^{=n}(n-1)i$, we have

$$\left(\frac{H_n(kr)}{H_n(k)}\right) \leq \left(\frac{H_n(2k)}{H_n(k)}\right)$$

$\leq 2^{-(n+1)} \; for \; all \; integer \; n \; greater \; thn \; some \; N \; for \; all \; r \geq 2$

Similarly, we have,

$$\left|k\left(\frac{H_{n,}'(kr)}{H_n(k)}\right)\right| \leq \left|k\left(\frac{H_{n+1}(2k)}{H_n(k)}\right)\right| + \left|\frac{n}{r}\left(\frac{H_n(kr)}{H_n(k)}\right)\right| \leq 2^{-(n+1)}$$

$for \; all \; integer \; n \; greater \; thn \; some \; N \; for \; all \; r \geq 2$

Given that, a_n is bounded, it follows that the series $\sum_{n=-\infty}^{\infty} ine^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)$ and its r derivatives are uniformly converges for all r in $[2,\infty]$

Finally, we need to show that the uniform convergent of $u_{rr} \; and \; u_{\theta\theta} \; for \; r > 1$.

Remarks:

Similar argument can be carried out for all second derivatives. Next, we recall that $H_n(r)$ satisfies the Bessel Differential Equations

$$y''(r) + \frac{1}{r}y'(r) + \left(1 - \frac{n}{r^n}\right)y(r) = 0$$

To argue that each function $e^{in\theta}H_n(kr)$ satisfies Helmholtz Equation due to the form of the Laplacian in Polar coordinates namely,

$$\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

All together this shows that the function defined by the series

$u = \sum_{n=-\infty}^{\infty} ine^{in\theta} k\left(\frac{H_n(kr)}{H_n(k)}\right)$ satisfies

$\Delta u + k^2 u = 0$ and the equation

$$\frac{\partial u}{\partial r} - iku = O(r^{-1})$$

It is added to the Dirichlet Neumann problem guarantee uniqueness.

3.5. Lemma – 3.6:

For any $n \; in \; N$, $|H_n(z)|$ is a decreasing function of $z \; in \; (0,\infty)$

Proof:

This is proved due to the formula derived by Nicholson Concept,

$$J_n^2(z) + Y_n^2(z) = \frac{8}{\pi^2}\int_0^\infty k_0 \, 2z\sinh(t)\cosh(2nt)\,dt,$$

$$where \; k_0(s) = \int_0^\infty e^{-s\cosh(t)dt}$$

Note that, for any fixed $r > 1$ and the series $u = \sum_{n=-\infty}^{\infty} ine^{in\theta} k \left(\frac{H_n(kr)}{H_n(k)}\right)$ *is* in the Sobolev space $H^{\frac{1}{2}}(\partial D)$ *for all* $r > 1$ and that, further application of the above formula we will show that the series convergence strongly to f *in* $H^{\frac{1}{2}}(\partial D)$ *as* $r \longrightarrow 1^{+1}$

Define the linear operator

$$T_k(f) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} k \left(\frac{H_n'(kr)}{H_n(k)}\right) \tag{3.29}$$

Where, $f = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$

Since T_k is continuous that is implies $k\left(\frac{H_n'(kr)}{H_n(k)}\right) \sim -n$ *for* $n \longrightarrow \infty$

According to lemma – 3.1, an equivalent way of defining T_k to say that, it maps f *to* $\frac{\frac{\partial u}{\partial r}}{r} = 1$ *where* u solution to the problem of lemma 3.1.

Denoted by $\langle . \rangle$ the duality bracket between $H_n^{\frac{1}{2}}$ *and* $H_n^{-\frac{1}{2}}$ and it extends to the dot product

$$\langle f, g \rangle = \int_{\partial D}^{.} f\overline{g}$$

3.6. Lemma 3.7:

T_k ii is real, analytic in k for k *in* $(0, \infty)$

Proof:

Let k_0 *and* b are two real numbers such that $0 < b < k_0$ and define $D_b(k_0)$ it is the closed disk in the complex plane centered at k_0 *with radious* b

To show that, T_k ii is real, analytic in k for k *in* $(0, \infty)$ in operator norm. it suffices to fix f *and* g *in* $H_n^{\frac{1}{2}}$

Simply to show that $Re\langle T_k(f), g\rangle$ it is an analytic function of k *in* $D_b(k_0)$

Take $f = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ *and* $g = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ and then we have

$$\langle T_k(f), g\rangle = 2\prod \sum_{n=-\infty}^{\infty} a_n e^{in\theta}\, \overline{b_n e^{in\theta}}\, k\left(\frac{H_n'(kr)}{H_n(k)}\right) \tag{3.30}$$

$$\text{Since } k\left(\frac{H_n'(kr)}{H_n(k)}\right) \sim k\left(-\frac{H_{n+1}'(kr)}{H_n(k)}\right) + \frac{n}{k}\left(\frac{H_n(kr)}{H_n(k)}\right) = -k\frac{H_{n+1}'(kr)}{H_n(k)} + n = -n$$

$$\Rightarrow k\left(\frac{H_n'(kr)}{H_n(k)}\right) \sim -n \Rightarrow \sum_{n-\infty}^{\infty} \left|a_n b_n k\left(\frac{H_{n+1}'(kr)}{H_n(k)}\right)\right| = \sum_{n-\infty}^{\infty} n\left|a_n \overline{b_n}\right| < \infty$$

And since, $k\left(\frac{H_n'(kr)}{H_n(k)}\right) \sim -|n|$ *for* $n \longrightarrow \infty$ uniformly for all k *in* $D_b(k_0)$

$\Rightarrow$the series is uniformly convergent sum of analytic function of k thus, $\langle T_k(f), g\rangle$ is analytic in $D_b(k_0)$

3.7: Lemma – 3.8:

The operator T_k converges strongly to the operator T_0 and it maps $H^{\frac{1}{2}}(\partial D)$ *into* $H^{-\frac{1}{2}}(\partial D)$ and it is defined by the formula

$$T_0(f) = \sum_{n=-\infty}^{+\infty} |n| a_n e^{in\theta} \tag{3.31}$$

Where $f = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$

Proof:

Since $f = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$ *in* $H^{\frac{1}{2}}(\partial D)$

Write, $\|T_k(f) = T_0(f)\|_{H^{-\frac{1}{2}}(\partial D)} = \left|\sum_{n=-\infty}^{+\infty} \frac{|a_n|^2}{\sqrt{n^2+1}} k\left(\frac{H_n^2(kr)}{H_n(k)}\right) - |n|^2\right|$ and then apply the lemma 3.3. and then we get required result.

3.9. Lemma – 3.9:

Let L it is a continuous linear function on $H_{0,\Gamma}(\Omega)$ and the following variational problem a unique solution and then find u *in* $H_{0,\Gamma}(\Omega)$ such that

$$\int_{\Omega}^{.} \nabla u . \nabla u - \int_{\partial D}^{.} (T_0 u) v = L(v) \text{ for all } v \text{ in } H_{0,\Gamma}(\Omega) \tag{3.32}$$

Proof;

We observe that, due to definition of $T_0, \langle T_0(f), f\rangle$ is real for all f in $H^{\frac{1}{2}}(\partial D)$ *and* $\langle T_0(f), f\rangle \leq 0$

We conclude that, it is uniquely solvable and that the solution u depends continuously on L

Let φ it is a smooth compactly supported function in D and it is equal to 1 on Γ and such that $\varphi(x_1, -x_2) = \varphi(x_1, x_2)$ *for* $k \geq 0$ and set $\overline{u_k}$ *in* $H_{0,\Gamma}(\Omega)$, it is the solution to

$$\nabla\overline{u_k}. \nabla v - k^2\overline{u_k} v - \int_{\partial D}^{.} (T_k \overline{u_k}) v = \int_{\partial D}^{.} (\nabla\varphi + k^2\varphi) v, \text{for all } v \text{ in } H_{0,\Gamma}(\Omega) \tag{3.33}$$

And we set $\overline{u_k} = u_k$ and then we get required result.

3.10. Lemma – 3.10:

For $k > 0$ *and* u_k satisfies the following properties

1. u_k *is in* $H'(\Omega)$
2. The upper and lower traces on Γ of u_k are both equal to the constant 1
3. u_k can extend to a function $\mathbb{R}^2/\Gamma$ such that if we still denote by u_k that extension

$$(\Delta + k^2) u_k = 0 \text{ in } \mathbb{R}^2/\Gamma$$

$$\frac{\partial u_k}{\partial r} - iku_k = O(r^{-1}) \text{ uniformly as } r \longrightarrow \infty$$

$$u_k(x_1, -x_2) = u_k(x_1, x_2) \text{ for all } (x_1, x_2) \text{ in } \mathbb{R}^2/\Gamma$$

$$\frac{\partial u_k}{\partial r}(x_1, 0) \text{ if } (0, x) \text{in } \Gamma$$

4. Denoted by $\frac{\partial u_k}{\partial x_2}$ it is the lower trace of $\frac{\partial u_k}{\partial x_2}$ *on* Γ and then

$$Im \int_{\Gamma}^{.} \frac{\partial u_k}{\partial x_2} \leq 0 \tag{3.34}$$

Proof:

Properties (1) and (2) are obviously true.

In Properties of (3) the first two conditions of properties of (3) holds simply because we can write

$\frac{u}{\partial D} = a_n e^{in\theta}\left(\frac{H_n(kr)}{H_n(k)}\right)$ $in\ \mathbb{R}^2/\overline{\Omega}$ and then using the lemma – 3.1. in the combination to the fact that variational problem implies that $T_k u_k$ is the limit of $\partial\left(\frac{\partial u_k}{\partial r}\right)$ $as\ r \longrightarrow 1^-$

To show that the third item in property (3) we set $\widetilde{u_k}(x_1, x_2)$ and for any arbitrary $v\ in\ H_{0,\Gamma}(\Omega)$

For $\underline{v}(x_1, x_2) = v(x_1, -x_2)$, next we observe that

$$\int_\Omega^{\cdot} \partial\underline{\widetilde{u_k}}.\nabla v - k^2\,\underline{\widetilde{u_k}}.v = \int_\Omega^{\cdot} \nabla\widetilde{u_k}.\nabla\underline{v} - k^2\widetilde{u_k}\underline{v}$$

In polar coordinates we have the relation $\widetilde{u_k}(r,\theta) = \widetilde{u_k}(r,-\theta)\ and\ \underline{v}(r,\theta) = \underline{v}(r,-\theta)$

$$\Rightarrow \int_{\partial D}^{\cdot}\left(T_k\underline{\widetilde{u_k}}\right)v = \int_{\partial D}^{\cdot}\left(T_k\widetilde{u_k}\right)\underline{v}$$

Finally, since, φ is even in $x_2 \Rightarrow \int_\Omega^{\cdot}(\Delta\varphi + k^2\varphi)\,\underline{v} = \int_\Omega^{\cdot}(\Delta\varphi + k^2\varphi)\,v$

Since the solution of the problem is unique, we must have $\underline{\widetilde{u_k}.} = \widetilde{u_k}$ proving the third item in (3)

since, u_k is even in $x_2 \Rightarrow \left(\frac{\partial u_k}{\partial x_2}\right)$ is zero on the line $x_2 = 0\ minus\ the\ segment\ \Gamma$, proving the last term in (3),

since $u = 1$ on Γ; $Im\int_\Gamma^{\cdot}\left(\frac{\partial u_k}{\partial x_2}\right)\overline{u_k} = Im\int_\Gamma^{\cdot}\left(\frac{\partial u_k}{\partial x_2}\right)u_k = -Im\int_D^{\cdot}\left(\frac{\partial u_k}{\partial r}\right)\overline{u_k}\ where\ \partial D$ it is the intersection of circle ∂D and the lower half plane $x_2 < 0$

We use parity one more time to argue that

$$m\int_D^{\cdot}\left(\frac{\partial u_k}{\partial r}\right)\overline{u_k} = \frac{1}{3}Im\int_{\partial D}^{\cdot}\left(\frac{\partial u_k}{\partial r}\right)\overline{u_k}$$

Note that, u_k it is not zero everywhere and using Redlich's lemma we can claim that

$Im\int_{\partial D}^{\cdot}\left(\frac{\partial u_k}{\partial r}\right)\overline{u_k} > 0$

3.11. Lemma – 3.11 (without proof)

Let $\overline{u_k}$ is the solution for all $k > -0\ and\ set\ u_k = \widetilde{u_k} + \varphi$ and then we have

1. u_k is analytic in $k\ doe\ k > 0$
2. u_k converges strongly to $u_0\ inH^1_{0,\Gamma}(\Omega)$morepreciously there is a constant C such that

$$\|u_k - u_0\|H^1_{\cdot}((\Omega) \leq C[k^2\|T_k - T_0\|] \tag{3.35}$$

3.12 Lemma – 3. 12

1. Consider $\left(\frac{\partial u_k}{\partial x_2^{\pm}}\right)$ the upper and lower traces of $\left(\frac{\partial u_k}{\partial x_2}\right)$ $on\ \Gamma$ and then

$$\left(\frac{\partial u_k}{\partial x_2^{\pm}}\right) = \left(\frac{\partial u_k}{\partial x_2^{-}}\right) \tag{3.36}$$

2. Consider, $G_k(x,y) = \frac{1}{4}H_0(k|x-y|)\ for\ all\ x\ in\ \Omega$ and then we have

1. Consider $\left(\frac{\partial u_k}{\partial x_2^{\pm}}\right)$ the upper and lower traces of $\left(\frac{\partial u_k}{\partial x_2}\right)$ $on\ \Gamma$ and then

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Consider, $G_k(x,y) = \frac{1}{4}H_0(k|x-y|)\ for\ all\ x\ in\ \Omega$ and then we have

$$u_k(x) = 2\int_\Gamma^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial x_2}\right)(y)dy \tag{3,37}$$

Proof:

Consider, $\Omega^+ = (x_1,x_2) \in \Omega, x_2 > 0$ and $\Omega^- = (x_1,x_2) \in \Omega, x_2, < 0$. It is well known from Potential Theory that if $x\ in\ \Omega^+$

$$u_k(x) = \int_{\partial\,\Omega^+}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n}\right)(y) - \int_{\partial\,\Omega^+}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n(y)}\right)u_k(y)dy \tag{3, 38}$$

And

$$0 = \int_{\partial\,\Omega^+}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n}\right)(y) - \int_{\partial\,\Omega^+}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n(y)}\right)u_k(y)dy \tag{3, 39}$$

Where n it is the exterior normal vector in each case. If y it is in $\partial\,\Omega^-$ and is such that $y_2 = 0$

It is clear that, $\left(\frac{G_k(x,y)}{\partial n(y)}\right) = 0$, we also use that u_k It is even in x_2 so that, $\left(\frac{\partial u_k}{\partial x_2}\right)(x) = 0$ if $x_2 = 0$ and x, it is not in Γ and since u_k it is even in x_2 and we can find that, for $x\ in\ \Omega^+$

$$u_k(x) = \left[\int_{\partial D}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n(y)}\right)(y) - G_k(x,y)\left(\frac{\partial u_k}{\partial n(y)}\right)u_k(y)\right]dy - \int_\Gamma^{\cdot} G_k(x,y)\,G_k(x,y)\left(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-}\right)(y)dy \tag{3.40}$$

But due to Lemma 3.10 Property (3) and since $u_k = \widetilde{u_k}\ on\ \partial D$ and then we have,

$$\int_{\partial D}^{\cdot} G_k(x,y)\left(\frac{\partial u_k}{\partial n}\right)(y) - G_k(x,y)\left(\frac{\partial u_k}{\partial n}\right)u_k(y) = 0\ \ for\ all\ x\ in\ \Omega^+$$

, that is why we get

$$(x) = \int_\Gamma^{\cdot} G_k(x,y)\,G_k(x,y)\left(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-}\right)(y)dy \tag{3.41}$$

For all $x\ in\ \Omega^-$

Take x_2 it is the derivative of $x\ in\ \Omega^+$ approaching Γ

Due to the normal derivatives of single layer potential, we find that

$$\left(\frac{\partial u_k}{\partial x_2^+}\right)(y) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-}\right)(x) - \int_\Gamma^{\cdot} G_k(x,y)\,G_k(x,y)\left(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-}\right)(y)dy$$

Observe that, for $x\ and\ y\ on\ \Gamma$ and hence completes the proof.

3.12 – Theorem:

The following estimates as $k\ approaches\ 0^+\ holds$

$$\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)\sim\pi k\left(\frac{H_1(k)}{H_0(k)}\right) \tag{3.42}$$

Consequently, $Re\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)$ must be strictly positive for small values of $k<0$

Proof:

Set $v=1-\varphi$ in variational problem (Note the trace of v is zero on Γ, as required in the space $H_{0,\Gamma}(\Omega)$ to obtain

$$-\int_{\Omega}\nabla\widetilde{u_k}.\nabla\varphi-\int_{\Omega}k^2\widetilde{u_k}.(1-\varphi)-\int_{\partial D}T_k\widetilde{u_k}.(1-\varphi)=\int_{\Omega}(\Delta\varphi+k^2\varphi)(1-\varphi) \tag{3.43}$$

First observe that,

$\int_{\Omega}k^2\,\varphi(1-\varphi)=O(k^2)$ and due to the lemma (3,11)

$\int_{\Omega}k^2\widetilde{u_k}\,(1-\varphi)=O(k^2)$ and then we have to find that

$$-\int_{\Omega}\nabla\widetilde{u_k}.\nabla\varphi+\int_{\Omega}\nabla\varphi=\int_{\partial D}T_k\widetilde{u_k}+O(k^2) \tag{3.44}$$

Using Green's Theorem, we get,

$$2\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)=2\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)\varphi=\int_{\Omega}\nabla\widetilde{u_k}.\nabla\varphi+\int_{\Omega}\Delta\widetilde{u_k}.\varphi \tag{3.45}$$

Since in Ω, $\Delta\widetilde{u_k}=-\Delta\,\varphi-k^2\widetilde{u_k}=k^2\varphi$ and it yields

$$2\int_{\Gamma}\left(\frac{\partial\widetilde{u_k}}{\partial x_2^-}\right)=-\int_{\partial D}T_k\widetilde{u_k}+O(k^2) \tag{3.46}$$

But $\int_{\partial D}T_k\widetilde{u_k}=-k\left(\frac{H_1(k)}{H_0(k)}\right)(2\emptyset)a_0(k)$ *where* $a_0=\frac{1}{2\pi}\int_{\partial D}\widetilde{u_k}$ so that using again that $\widetilde{u_k}$is strongly convergent to $(1-\varphi)$ *in* $H^1(\Omega)$, *and also* $a_0(k)$*tends to* 1 *as* $k\longrightarrow 0$

We claim that

$$\int_{\partial D}T_k\widetilde{u_k}\sim-2\pi k\left(\frac{H_1(k)}{H_0(k)}\right)(k) \tag{3.47}$$

as $k\longrightarrow 0$. Again, going back to the definition of Hankel function it is easy to see that

$k\left(\frac{H_1(k)}{H_0(k)}\right)\sim-(ln^k)^{-1}$ *as* $k\longrightarrow 0$, from this we conclude that

$$2\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)\sim-2\pi k(ln^k)^{-1} \tag{3.48}$$

So that we have $Re\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)$must be strictly positive for all $k>1$ as small enough/

Hence completes the proof of theorem.

4. ASYMPTOTIC FOR HIGH WAVE NUMBER

We provide in this section – 4; a derivation of an equivalent for $\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)$ *as* $k\longrightarrow\infty$ *where* $u_k=\widetilde{u_k}+\varphi$ *and* $\widetilde{u_k}$ solve the variational problem. We will prove the following theorem

4.1. Theorem

Let $\widetilde{u_k}$ it is a solution and set $u_k=\widetilde{u_k}+\varphi$, the following estimates hold as $k\longrightarrow\infty$ we have

$$\int_{\Gamma}\left(\frac{\partial u_k}{\partial x_2^-}\right)=-ik+O\left(k^{\frac{3}{4}}\right) \tag{4.1}$$

Before we will prove the theorem 4.1. we will develop some theory in the following manner"

What is the asymptotic behavior of $\left(\frac{\partial u_k}{\partial x_2^-}\right)$ as the wave number $k\longrightarrow\infty$?

High frequency approximation for the wave equation is a vast subject which has been extensively studied over time. Historically, investigations have tried to explain how the laws of geometric optics relate to the wave equation at high frequency in an attempt to provide a sound foundation for Fresnel's law, Kirchhoff may have been the rest one to write specific equations and asymptotic formulas for high frequency wave phenomena, however, his derivation was informal. More mathematically rigorous study of the behavior of solutions to the wave equation at high frequency requires the use of Fourier Integral Operators and Micro – Local Analysis. As far as we know this kind of work was pioneered by Majda Melrose and Tayler (2012). These authors were actually interested in the case of exterior of a bounded convex domain. So their results cannot be applied to our case since, has empty interior in $\mathbb{R}^2$, We have instead to rely on recent ground breaking work by Hewett , Langdon, and Chandler – Wilde, it pertains to either scattering in dimension – 2 by soft or hard line segments (/This is our study) or scattering in dimension – 3 by soft or hard open planner surfaces. The great achievement of those above authors work is that they were able to derive continuity coercivity bounds that explicitly depend on the wave number.

Following the work by Hewett and Chandler – Wilde we introduce relevant function spaces and frequency depending norms. Let v *on* $\mathbb{R}$ *and* $\widehat{v}$its Fourier Transform. Let s *on* $\mathbb{R}$, we say that v *is in* $H^s(\mathbb{R})$, if $\left(\int_{\mathbb{R}}(1+\xi^2)^s\,|\widehat{v}(\xi)^2|d\xi\right)^{\frac{1}{2}}\leq\infty$

And then we define in $H^s(\mathbb{R})$ *and the* k is dependent norm so that we have

$$\|v\|H_k^s(\mathbb{R})=\left(\int_{\mathbb{R}}(k^2+\xi^2)^s\,|\widehat{v}(\xi)^2d\xi|\right)^{\frac{1}{2}} \tag{4.2}$$

Note that, $H^1(\mathbb{R})$ is included in $H^{\frac{1}{2}}(\mathbb{R})$, and more preciously

$$\|v\|_{H_k^{\frac{1}{2}}(\mathbb{R})}\leq k^{-\frac{1}{2}}\|v\|_{H_k^{\frac{1}{2}}(\mathbb{R})} \tag{4.3}$$

For all v *in* $H^1(k)$. Let us take an interval $I=\left(-\frac{1}{2},\frac{1}{2}\right)$. $\widetilde{H^s}$ is defined to be the closure of $C_e^{\infty}(I)$

(The space of smooth functions, compactly supported in I). For the norm $\|v\|_{H_k^{\frac{1}{2}}(\mathbb{R})}$,$H^s(I)$ is defined to be the space of restrictions to I of elements in $H^s(\mathbb{R})$, we define on $H^s(I)$, the norm

$$\|v\|_{H_k^{\frac{1}{2}}(\mathbb{R})} = \inf\left\{\|v\|_{H_k^{\frac{1}{2}}(\mathbb{R})} : v \in \|v\|_{H_k^s(\mathbb{R})}\ and \frac{v}{I} = v\right\}$$

4.2. Hewett and Chandler – Wilde – Theorem (Without Proof)

For any set $s\ in\ \mathbb{R}$ the operator S_k defined by the following formula for smooth function $v\ in\ I$ then $\|S_k v(x_1)\| \approx \int_\Gamma^{\cdot} \frac{1}{4} H_0(k|x_1 - y_1|)\, v(y_1) dy_1$, it can be extended to a combination of linear operator from $\widehat{H^s}\ (I)\ to\ H^{s+1}(I)$, Furthermore, S_k is injective, S_k^{-1}continuous and satisfies the estimate

$$\|S_k^{-1} v\|_{H_k^{\frac{1}{2}}(I)} \leq 2\sqrt{2}\|v\|_{H_k^{\frac{1}{2}}(I)} \quad (4.4)$$

For all $v\ in\ H_k^{\frac{1}{2}}(I)$

Let us emphasize one more time that although the continuously and coercivity properties of S_k known for some time. Hewett and Chandler – Wilde's great achievement was to derive the dependency of the coercivity bounds on the wave number $k\ \ as$ in estimate, the dependency =appears in the use of special norms $\|, \|_{H_k^s(I)}$

4.3. An Informal Derivation of Estimation

This informal derivation will be helpful, since it will give us idea of what the asymptotic behavior of

for $\int_\Gamma^{\cdot} \left(\frac{\partial u_k}{\partial x_2^-}\right)(0, y_1)\ for\ y_0 = 1$

Since f_k satisfies the integral equals to $S_k f_k = \frac{1}{2}$ Or

$$\int_I^{\cdot} \frac{1}{4} H_0\left(k|x_1 - y_1| f_k(y_1)\right) dy_1 = \frac{1}{2};\ x_1 \in I \quad (4.5)$$

Multiply above equation by $-2ik$ and integrate in x_1 over I, we get

$$\int_I^{\cdot} \int_I^{\cdot} \frac{k}{4} H_0\left(k|x_1 - y_1| f_k(y_1)\right) dy_1 dx_1 = -ik \quad (4.6)$$

If possible, interchange order of integration in the left =hand side of above equation, then

$$\int_I^{\cdot} \int_I^{\cdot} \frac{k}{4} H_0\left(k|x_1 - y_1| dx_1 f_k(y_1)\right) dy_1 = -ik \quad (4.7)$$

Now, for every $x_1\ in\ I$, setting $v = k(x_1 - y_1)$ and using the following (lemma – 4.1) we can find that

$$\lim_{k\to\infty} \int_I^{\cdot} \frac{k}{2} H_0\left(k|x_1 - y_1| dx_1\right) = 1 \quad (4.8)$$

So that, we led to, believe that,

$\int_I^{\cdot} f_k(y_1) y_1 \sim -ik$, it would set out to prove in future.

4.4. Lemma – 4.1

Write St_n, the Struve function of order n and then the following condition holds for any $t > 0$

$$\int_0^t H_0(z)\, dz = tH_0(t) + \frac{\pi}{2} t[St_0 H_1(t) - St_1 H_0(t)] \quad (4.9)$$

It follows that semi – convergent integral $\int_0^\infty H_0(z) dz$ is exactly equal in 1.

Proof:

Since the integral formula given the value $\int_0^\infty H_0(z) dz$ of the results from that formula combined with known asymptotic at infinity of Bessel and of Struve functions. One should consult for formula on Bessel Functions, and their derivatizations and we will get the results.

4.5. Rigorous Derivation of Estimate

The following estimate holds as k as approaches infinity

$$\left\|S_k\left(-ik - \frac{1}{2}\right)\right\|_{H_k^{\frac{1}{2}}(I)} = O\left(k^{\frac{1}{4}}\right) \quad (4.10)$$

Proof:

First, we start by recalling the above lemma (4.1) and noting that for $x_1\ in\ I$ and then,

$$S_k(-ik)(x_1) - \frac{1}{2} = \int_I^{\cdot} \frac{1}{4} H_0\left(k|x_1 - y_1| dy_1\right) - \frac{1}{2}$$

$$= -\int_{k\left(\frac{1}{2}+x_1\right)}^{\cdot} \frac{1}{4} H_0(v)\, dv$$

$$- \int_{k\left(\frac{1}{2}-x_1\right)}^{\cdot} \frac{1}{4} H_0(v)\, dv$$

Now set $G_k(x_1) = \int_{k\left(\frac{1}{2}+x_1\right)}^{\cdot} \frac{1}{4} H_0(v)\, dv - \int_{k\left(\frac{1}{2}-x_1\right)}^{\cdot} \frac{1}{4} H_0(v)\, dv$ and since $t \longrightarrow \int_t^\infty H_0(v) dv$ is continuous on $(0, \infty)$ and has limit zero at infinity, there is a positive C such that

$$\left|\int_t^\infty H_0(v) dv\right| \leq C \quad (4.11)$$

For $in\ (0, \infty)$. It is well known that v approaches infinity. Now we have

$H_0(v) e^{i\left(v - \frac{1}{4}\right)} \sqrt{\frac{2\pi}{v}} + O\left(v^{-\frac{3}{2}}\right)$, so we also have the estimate

$$\left|\int_t^\infty H_0(v) dv\right| = O\left(t^{-\frac{1}{2}}\right)\ as\ t \longrightarrow \infty \quad (4.12)$$

Without loss of generality, we nay assume that $k > 4$. If $x_1\ is\ in\ \left(-\frac{1}{2} + k^{\frac{1}{2}}, \frac{1}{2} - k^{\frac{1}{2}}\right)$ and then $k\left(\frac{1}{2} + x_1\right)\ and\ k\left(\frac{1}{2} - x_1\right)$ are greater than $k^{\frac{1}{2}}$, so that we have

$G_k(x_1) = O\left(k^{-\frac{1}{4}}\right)$ and then we infer that

$$|G_k|^2 = \int_{\left(-\frac{1}{2}+k^{\frac{1}{2}}, \frac{1}{2}-k^{\frac{1}{2}}\right)}^{\cdot} |G_k|^2 + \int_{\left(-\frac{1}{2}+k^{\frac{1}{2}}, \frac{1}{2}-k^{\frac{1}{2}}\right)}^{\cdot} |G_k|^2 = O\left(k^{-\frac{1}{2}}\right) \quad (4.13)$$

Next, note that, $G_k'(x_1) = kH_0\left(k\left(\frac{1}{2} + x_1\right) - kH_0 k\left(\frac{1}{2} - x_1\right)\right)$

Using substitution, we find that

$$\int_I |G_k'(x_1)|^2 \leq 4k \left(\int_0 H_0(v) \right)^2 dv$$

Given that asymptotic at infinity of H_0 we infer that

$$\int_I |G_k'(x_1)|^2 = O(klnk) \; as \; k \longrightarrow \infty \tag{4.14}$$

Now evaluate a few Sobolov norms of G_k.We observe that

$$G_k\left(-\frac{1}{2}\right) = G_k\left(\frac{1}{2}\right) = \int_0^{\infty} H_0(v)\, dv + \int_k^{\infty} H_0(v)\, dv$$

Uniformly bounded in k

4.6. Proof of Theorem – 4.1.

Case – 1: For $C_1, C_2, C_3, C_4, > 0$

In this case – 1, we cover the case where the constants C_1, C_2, C_3, C_4 are all positive, which was assumed in this theorem 4.1. We explained how the function denied relates to $u_k = \tilde{u}_k + \varphi$ And solve and it is extended to $\overline{\mathbb{R}^2}$ as indicated by lemma – 3.10 and then $u_k = \varphi \; in \; \overline{\mathbb{R}^2}$

Recalling the definition of the function F and it expressed as

$$F(k) = q(k^2) + p(k^2) Re\left(\int_\Gamma \left(\frac{\partial u_k}{\partial x_2^-} \right) \right) \tag{4.15}$$

We know that the lemma – 3.11 that F is analytic in $(0, \infty)$, again recalling the definition of $p \; and \; q$ and using the two estimates and then we get

$$F(k) \sim C_2 \pi (\ln k)^{-1} \; {}^{for} \; k \longrightarrow 0^+ \tag{4.16}$$

Again, recalling the two estimates, we claim that

$$F(k) \sim C_3 k^4 \;\; for \; k \longrightarrow \infty \tag{4.17}$$

It follows that, by the two estimates there is a positive α such that, $F(k) < 0$ if $k \in (0, \infty)$ and also, we infer that then we conclude that $\lim_{k \to \infty} F(k) = \infty$, since F is continuous in $(0, \infty)$, we conclude that F must achieve that value zero in that interval. We claim that the zeros of F are isolated since F is an analytic function and these zeros occur in some interval $[A, B]$ where A, B are two positive constants. In particular the equation $F(k) = 0$ has at most a finite number of solutions.

Case – 2: Use of estimate in the case where $C_3 < 0$

We must claim that for any positive values C_1, C_2, C_4 there exists a negative value of C_3 , then the equation $F(k) = 0$ has no solution. According to estimate there exists a positive number $k \; in \; (0, \infty)$ such that

$$-\frac{1}{2}\pi(\ln k)^{-1} \leq \; Re\left(\int_\Gamma \left(\frac{\partial u_k}{\partial x_2^-} \right) \right) \leq \frac{3}{2}\pi(\ln k)^{-1} \tag{4.18}$$

Consequently for $k \; in \; (0, \infty)$

$$F(k) \leq C_4 k^2 + C_1 \left(-\frac{3}{2}\pi(\ln k)^{-1} \right) + C_2 \frac{1}{2}\pi(\ln k)^{-1} \tag{4.19}$$

So that, if α is small enough, $F(k) < 0, and \; if \; k \;\; is \; in \; [0, \infty)$, there is negative A_1 and a positive A_2 for all $k \; in \; (0, \infty)$ such that

$$A_1 k^{\frac{3}{4}} \leq \; Re\left(\int_\Gamma \left(\frac{\partial u_k}{\partial x_2^-} \right) \right) \leq A_2 k^{\frac{3}{4}} \tag{4.20}$$

So that, for all $k \;\; is \; in \; [\alpha, \infty)$ and then we have

$$F(k) \longrightarrow C_3 k^4 + C_4 k^2 + C_1 A_2 k^{\left(2+\frac{3}{4}\right)} - C_1 A_1 k^{\frac{3}{4}} \tag{4.21}$$

So that, if we choose C_3 less that that some negative constant $F(k) < 0$ for all $k \; in \; [0, \infty)$. We conclude that for that choice of C_3 for all k, so that the equation $F(k) = 0$ has no solution in $(0, \infty)$.

Next, we show that the following claim for any value of the positive constants C_1, C_2 and any negative constant then the equation $F(k) = 0$ has at least one solution. We may assume that, α defined as above is less than 1, for all k in $(0, \infty)$, we get

$$F(k) \geq C_3 k^4 + C_4 k^2 + C_1 A_2 k^{\left(2+\frac{3}{4}\right)} - C_1 A_1 k^{\frac{3}{4}} \tag{4.22}$$

Thus, $F(k) < 0$ for any C_4 greater than some constant. As$C_3 < 0$, we have that $\lim_{k \to \infty} C_3 = -\infty$/ We conclude that the equation $F(k) = 0$ has at least one solution.

Hence completes the proof of Theorem.

5. CONCLUSION AND PERSPECTIVES

In this article the well – pawedness of a set of equations modeling City – Effect was powerful. To the best of our knowledge, up to now, there was no formal proof of existence of frequency between earth and vibration of Tall Building. Recalling that our research pertains exclusively to Ant – Plane shearing. It will be important to generalize our results to fully three – dimensional Elastic Vibration. We believe that only minor modification to our present argument, we ensured that the existence of frequency between the Earth and Tall Building

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