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Rellich-Kondrakov Embedding of the Laplacian Resolvent on the Torus

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Abstract: This paper proves that the domain of the Laplacian, Δ, on a closed Riemannian manifold, (M,g), is compactly embedded in $L^2(M)$. Particularly, the resolvent of the Laplacian, $(\Delta+1)^{-1}$, is shown to be compactly embedded on the torus.

Keywords: Laplacian; Resolvent; Sobolev space; Compactly embedding; Riemannian manifold; Torus.
MSC class: 35P20; 35R01.

1. Introduction

The concept of Sobolev spaces are well understood on some compact and complete Riemannian manifolds and on locally compact groups such as the Heisenberg group, see e.g. [7, 8, 9, 13] and [14]. In this paper, we are interested in Sobolev spaces in a general setting as a tool for a better understanding of pseudodifferential operators such as the Laplacian on Riemannian manifolds, [1, 12]. We shall show that the resolvent of the Laplacian is compactly embedded on the torus.

We begin by defining the Sobolev space of integer order on smooth Riemannian manifolds and proceed to review the Sobolev embedding theorem on manifolds. Afterwards, we present the so-called Rellich-Kondrakov theorem on smooth compact torus which is the main result of the paper.

2. Materials and Methods

We gather the concepts of Sobolev space, Sobolev embedding theorems especially for Riemannian manifolds and the Rellich-Kondrakov theorem in this section as tools for the proof of compact embedding theorems of the Laplacian resolvent in the next section. Proofs of some of the theorems may be presented for purpose of completeness.

2.1 Sobolev Space on Riemannian manifold

To begin with, let Ω be an open subset of $\mathbb{R}^n$ and k an integer; $p \geq 1$ a real. Let $u : \Omega \to \mathbb{R}$ be a real-valued smooth function. Following the works of [1, 10, 2, 6, 5, 3], let

$$\|u\|_{k,p} := \sum_{0\leq|\alpha|\leq k} [\int_\Omega |D^\alpha u|^p \, dx]^{1/p} \tag{1}$$

where the distributional derivatives

$$D^\alpha u = (D^{\alpha_1} \cdots D^{\alpha_n})u; \; D_j = \frac{1}{i}\frac{\partial}{\partial x_j};$$

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and α is a multi-index. We define the following Sobolev spaces.

- H_k^p is the completion of $\{u \in C^\infty(\Omega) : \|u\|_{k,p} < \infty\}$ and
- $W_k^p = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega); |\alpha| \le k\}$.

Theorem 2.1 *[1]. For any Ω, k integer and $p \ge 1$; $H_k^p = W_k^p$.*

Now let (M, g) be a smooth Riemannian n-dimensional manifold. For an integer k and

$$u : M \to \mathbb{R}$$

smooth, we denote by $\nabla^k u$ the k^{th} covariant derivative of u and by $|\nabla^k u|$ its norm. Specifically,

$$\nabla^0 u = u;$$

$$\nabla^1 u = (\nabla u)_i = \partial_i u \equiv \frac{\partial u}{\partial x_i};$$

$$\nabla^2 u = (\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^n \partial_i (\sqrt{|g|}) g^{ij} \partial_j u;$$

$$\nabla^3 u = (\nabla^2 \nabla u)_{ijk} = \partial_i [\partial_{jk} - \Gamma_{jk}^l \partial_l] u$$

and so on; where we recognize $\nabla^2 = \Delta$ as the Laplacian on the manifold (M, g) and

$$\Gamma_{ij}^k := \frac{1}{2} g^{mk} [\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}]$$

is the Christoffel symbol, see e.g [11, 6]. Suppose for now that $M = \mathbb{R}^n$ with the Euclidean metric, we have

$$\nabla^2 u = (\nabla^2 u)_{ij} = \sum_{i=1}^n \partial_i^2 u.$$

Besides, in local coordinates, the norm of $\nabla^k u$ is expressed as

$$\|\nabla^k u\| = g^{i_1 j_1 \cdots i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}.$$

To define Sobolev space on (M, g), we set

$$A_k^p(M) = \{u \in C^\infty(M) : \int_M |\nabla^j u|^p \, dv(g) < \infty; j = 0, \cdots, k\}. \quad (2)$$

For $u \in A_k^p(M)$, we have

$$\|u\|_{H_k^p} = \sum_{j=0}^k [\int_M |\nabla^j u|^p \, dv(g)]^{1/p} \quad (3)$$

where $dv(g) = \sqrt{\det(g_{ij})} dx$ and the Lebesgue volume measure of $\mathbb{R}^n$ is dx.

Definition 2.2 *The Sobolev space $H_k^p(M)$ is the completion of $A_k^p(M)$ with respect to $\|\cdot\|_{H_k^p}$ defined in (3).*

One can look at $H_k^p(M)$ as a subspace of $L^p(M)$ where for $u \in L^p(M)$, we write

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$$\|u\|_p = (\int_M |u|^p)^{1/p}.$$

Also observe that when $p = 2$, then

$$\|u\|_{H_k^2} = \sum_{j=0}^{k} [\int_M |\nabla^j u|^2 \, dv(g)]^{1/2}$$

with the associated inner product $\langle u, v \rangle = \sum_{j=0}^{k} \int_M \langle \nabla^j u, \nabla^j v \rangle dv(g)$.

Definition 2.3 *[6]. A real-valued function u on M is called a Lipschitz function (or Lipschitzian) if there exists a constant $c > 0$ such that for $x, y \in M$, $|u(y) - u(x)| \leq c\, d_g(x, y)$.*

We now look at smooth Riemannian manifolds. The next theorem is very essential in this regard.

Theorem 2.4 *[2] Let (M, g) be a smooth Riemannian n-dimensional manifold and let*

$u: M \to \mathbb{R}$ be a Lipschitz function on M with compact support. Then, $u \in H_1^p(M)$ for any $p \geq 1$. In particular, if M is compact, any Lipschitz function in M is also in $H_1^p(M)$ too.

The property of Sobolev embedding on smooth Riemannian n-dimensional manifold is summarised in the next theorem. We follow Aubin [2] and Hebey [10] to present a proof of the next theorem here for a purpose of completion. We denote the space of test functions on a Riemannian manifold M by $\mathcal{D}(M) := C_0^\infty(M)$ and its dual space by $\mathcal{D}'(M)$.

Theorem 2.5 *Given that (M, g) is a smooth n-dimensional complete Riemannian manifold, then $\mathcal{D}(M)$ is dense in $H_1^p(M)$ for any $p \geq 1$.*

Proof. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 1 - t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

Let $u \in A_1^p(M)$; $p \geq 1$ real. For $x, y \in M$, set $u_j(y) = u(y) f(d_g(x, y) - j)$ where d_g is the distance associated to g and $j \in \mathbb{Z}$. By theorem 2.4 above, $u_j \in H_1^p(M)$ for any j. Now, since $u_j = 0$ outside any compact set $\Omega \subset M$, we have that for any j; u_j is the limit in $H_1^p(M)$ of some sequence of functions in $\mathcal{D}(M)$. So, if

$$(u_m) \in A_1^p(M) \to u_j \in H_1^p(M)$$

and if $\alpha \in \mathcal{D}(M)$, then,

$$(\alpha u_m) \in A_1^p(M) \to \alpha u_j \in H_1^p(M).$$

Consequently, we choose $\alpha \in \mathcal{D}(M)$ such that $\alpha = 1$ where $u_j \neq 0$. So independently, we have for any j,

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$$(\int_M |u_j - u|^p \, dv(g))^{1/p} \le (\int_{M\setminus B_r(j)} |u|^p \, dv(g))^{1/p}$$

and

$$(\int_M |\nabla(u_j - u)|^p \, dv(g))^{1/p} \le (\int_{M\setminus B_r(j)} |\nabla u|^p \, dv(g))^{1/p} + (\int_{M\setminus B_r(j)} |u|^p \, dv(g))^{1/p}.$$

Hence, $(u_j) \to u \in H_1^p(M)$ as $j \to \infty$. That is to say, u is the limit in $H_1^p(M)$ of some sequence in $\mathcal{D}(M)$.

2.2 Sobolev Embedding

Given two normed vector spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ with $E \subset F$, we say that E is continuously embedded in F denoted by $E°F$, if there exist some constant $c > 0$ such that for any $x \in E$, we have $\|x\|_F \le c\|x\|_E$. We demonstrate this concept for compact smooth Riemannian n-dimensional manifolds. First, note that the embedding is said to be compact if bounded subsets of $(E, \|\cdot\|_E)$ are pre-compact (relatively compact) in $(F, \|\cdot\|_F)$. Clearly, if the embedding $E°F$ is compact, it is continuous; see e.g. [1, 2, 15, 10, 11].

Theorem 2.6 *Let (M, g) be a compact smooth Riemannian n-dimensional manifold. For any real numbers $1 \le q < p$ and integers $0 \le m < k$ satisfying $\frac{1}{p} = \frac{1}{q} - \frac{k-m}{n}$, we have*

$$H_k^q(M) \to H_m^p(M).$$

Proof. It is enough to prove that $H_1^1(M) \to L^{n/(n-1)}$ is valid since for

$$p > 1, H_k^q(M) \to H_1^1(M).$$

Now since M is compact, it can be covered by a finite number of charts $(\mathcal{U}_m, \psi_m)_{m=1,\cdots,N}$ such that for any m, the components g_{ij} of g in $(\mathcal{U}_m, \psi_m)$ satisfy

$$\frac{1}{2}\delta_{ij} \le g_{ij} \le 2\delta_{ij}$$

as bilinear forms where δ_{ij} is the usual metric on $\mathbb{R}^n$. The constant $c = 2$ can be chosen for convenience.

Again, let (η_m) be a smooth partition of unity subordinate to $(\mathcal{U}_m)$. For any $u \in C^\infty(M)$, we have

$$\int_M |\eta_m u|^{n/(n-1)} \, dv(g) \le 2^{n/2} \int_{\mathbb{R}^n} |(\eta_m u) \circ \psi_m^{-1}(x)|^{n/(n-1)} \, dx$$

and

$$\int_M |\nabla(\eta_m u)|^{n/(n-1)} \, dv(g) \ge 2^{(n+1)/2} \int_{\mathbb{R}^n} |\nabla((\eta_m u) \circ \psi_m^{-1})(x)| \, dx.$$

Hence,

$$(\int_{\mathbb{R}^n} |(\eta_m u) \circ \psi_m^{-1}(x)|^{n/(n-1)} \, dx)^{(n-1)/n} \le \frac{1}{2} \int_{\mathbb{R}^n} |\nabla((\eta_m u) \circ \psi_m^{-1})(x)| \, dx$$

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for any m. Consequently, for any $u \in C^{\infty}(M)$,

$$(\int_M |u|^{n/(n-1)}\, dv(g))^{(n-1)/n} \le \sum_{m=1}^{N} (\int_M |\eta_m u|^{n/(n-1)}\, dv(g))^{(n-1)/n}$$

$$\le 2^{n-1} \sum_{m=1}^{N} \int_M |\nabla(\eta_m u)|\, dv(g)$$

$$\le 2^{n-1} \int_M |\nabla u|\, dv(g) + 2^{n-1} (\max_M \sum_{m=1}^{N} |\nabla \eta_m|) \int_M |u|\, dv(g)$$

which gives the result.

2.3 Rellich-Kondrakov theorem

Now, we are set to present the Rellich-Kondrakov theorem (which may also be called Rellich theorem for simplicity).

Theorem 2.7 *Let* (M, g) *be a compact smooth Riemannian* n*-dimensional manifold. For any integers* $j \ge 0; m \ge 1;$ *and real numbers* $q \ge 1, p$ *such that* $1 \le p \le \frac{nq}{n - mq}$, *the embedding*

$$H^q_{j+m}(M) \to H^p_j(M)$$

is compact. In particular, for any $q \in [1, n)$ real and any $p \ge 1$ such that $\frac{1}{p} > \frac{1}{q} - \frac{1}{n}$, the embedding $H^q_1(M) \to L^p(M)$ is compact.

To prove the Rellich theorem, we need the following lemma.

Lemma 2.8 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n; q \in [1, n)$ *real such that* $\frac{1}{p} > \frac{1}{q} - \frac{1}{n}$. *Then, the embedding* $H^q_{0,1}(\Omega) \to L^p(\Omega)$ *is compact; where* $H_{0,1}(\Omega)$ *denotes the closure of* $\mathcal{D}(\Omega) \subset H^q_1(\Omega)$.

For proof, one can see Aubin [2]. Next is the proof of theorem 2.7.

Proof. Since M is compact, it can be covered by a finite number of charts $(U_s, \psi_s)_{s=1,\cdots,N}$ such that for any s, the components g^s_{ij} of g in (U_s, ψ_s) satisfy $\frac{1}{2}\delta_{ij} \le g^s_{ij} \le 2\delta_{ij}$ as bilinear forms. Let (η_s) be a smooth partition of unity subordinate to the covering (U_s). Given (f_m) a bounded sequence in $H^q_1(M)$ and for any s, we let $f^s_m = (\eta_s f_m) \circ \psi_s^{-1}$. Clearly, (f^s_m) is a bounded sequence in $H^q_{0,1}(U_s, \psi_s))$ for any s.

By lemma 2.8, one then gets that a subsequence $(\tilde{f}^s_m)$ of (f^s_m) is a Cauchy sequence in $L^P(\psi_s(U_s))$.

Coming back to the inequality satisfied by the g^s_{ij}, one gets for any s, $(\eta_s f_m)$ is a Cauchy sequence in $L^p(M)$. But for any m_1, m_2,

$$\| f_{m_1} - f_{m_2} \|_p \leq \sum_{s=1}^{N} \| \eta_s f_{m_1} - \eta_s f_{m_2} \|_p ;$$

where $\| \cdot \|_p$ denotes the L^p-norm. Hence, (f_m) is a Cauchy sequence in $L^p(M)$. This proves the theorem.

Corollary 2.9 *The embedding* $H_0^1(M) \cap H_1^2(M) \to H_1^1(M) \to L^2(M)$ *is compact.*

Proof. Follows from the Rellich-Kondrakov theorem proved in lemma 2.8.

3. Results and Discussion

The compactness of $(\Delta+1)^{-1}$ on (M, g) is more conveniently discussed on Fourier space, [15, 3] and [4]. So we continue the discussion on the Sobolev space. The Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : (1+|\xi|^2)^s |\hat{u}|^2 \in L^2(\mathbb{R}^n)\}$$

where $\hat{u}$ is the Fourier transform of u. This means that a function $u \in H^s(\mathbb{R}^n)$ provided $\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty$ with

$$\| u \|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Now, define $\mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n = (S^1)^n$ to be the n-dimensional unit torus. A function $u \in \mathbb{T}^n$ can be expressed via Fourier series as

$$u(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{u}(\xi) e^{i\xi \cdot x} \text{ where } \hat{u} = (2\pi)^{-n} \int_{\mathbb{T}^n} u(x) e^{-i\xi \cdot x} dx.$$

For any function $u \in \mathcal{D}'(\mathbb{T}^n)$, we can write

$$\hat{u} = (2\pi)^{-n} \langle u(x), e^{-i\xi \cdot x} \rangle$$

so that by Plancherel theorem

$$\sum_{\xi \in \mathbb{Z}^n} |\hat{u}(\xi)|^2 = (2\pi)^{-n} \int_{\mathbb{T}^n} |u(x)|^2 \, dx$$

and

$$D^\alpha u(x) = \sum_{\xi \in \mathbb{Z}^n} \xi^\alpha \hat{u}(\xi) e^{i\xi \cdot x}.$$

So, it is now clear that $u \in C^\infty(\mathbb{T}^n)$ if and only if $\hat{u}$ is a rapidly decreasing function in $\mathbb{Z}^n$. That is, for $s \in \mathbb{R}$,

$$\sup_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty.$$

By duality, $u \in \mathcal{D}'(\mathbb{T}^n)$ provided $\hat{u}$ is polynomially bounded function, i.e.

$$|\hat{u}| = c(1+|\xi|^2)^l$$

for $c > 0, l \in \mathbb{R}$.

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Let T defined by $T=(\Delta+1)^{-1}$ be the Laplacian resolvent on $L^2(\mathrm{T}^n)$ where we know that

$$\mathrm{F}: L^2(\mathrm{T}^n)\to l^2(\mathrm{T}^n) \text{ and}$$
$$T=\mathrm{F}^{-1}(1+|\xi|^2)^{-1}\mathrm{F};$$
$$\Rightarrow \|T\|=1.$$

The next theorem is the main result of this work.

Theorem 3.1 *T is a compact operator on $l^2(\mathrm{T}^n)$.*

Proof. It enough to show that T is the limit of finite rank operators. To do this, define

$$(T_N u)(\xi)=\begin{cases}(1+|\xi|^2)^{-1}u_\xi & ;|\xi|\le N\\ 0 & ;|\xi|>N.\end{cases}$$

for $N>0$ and $u_\xi\in l^2(\mathrm{T}^n)$. Then,

$$(T-T_N)\psi_\xi=\begin{cases}(1+|\xi|^2)^{-1}\psi_\xi & ;|\xi|\ge N+1\\ 0 & ;|\xi|\le N.\end{cases}$$

So,

$$\|T-T_N\|=\frac{1}{(N+1)^2+1}.$$

But,

$$(T-T_N)\psi_\xi\le\frac{1}{(N+1)^2+1}\|\psi_\xi\|$$

$$\Rightarrow \Big(\sum_{\xi\in Z^n}|(1+|\xi|^2)^{-1}\psi_\xi|^p\Big)^{1/p}\le\frac{1}{(N^2+1)^p}\sum_{\xi\in Z^n}|\psi_\xi|^{1/p}$$

$$=\frac{1}{N^2+1}\sum_{|\xi|\ge N+1}|\psi_\xi|^p$$

$$\le\frac{1}{N+1}\|\psi_\xi\|_{l^p}\to 0 \text{ as N}\to\infty.$$

Therefore, $T=(\Delta+1)^{-1}$ is compact in $l^2(\mathrm{T}^n)$.

Another result of the Rellich-Kondrakov theorem for the torus is the next theorem.

Theorem 3.2 *Suppose T is a self-adjoint operator (or has self-adjoint extension) with compact resolvent, then T has discrete spectrum.*

Proof. It is straightforward to see that

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$$(T-\mathrm{id})^{-1}\psi_k = \mu_k\psi_k$$
$$\Rightarrow \psi_k = \mu_k(T-\mathrm{id})\psi_k$$
$$\Rightarrow \psi_k + \mu_k\psi_k = \mu_k T\psi_k$$
$$\Rightarrow T\psi_k = \frac{(\mathrm{id}+\mu_k)}{\mu_k}\psi_k.$$

Set

$$\lambda_k = \frac{(\mathrm{id}+\mu_k)}{\mu_k}$$

and write

$$T\psi_k = \lambda_k\psi_k.$$

We see that as $\mu_k \to 0$, $\lambda_k \to \infty$ which confirms the compactness of the resolvent $(T-\mathrm{id})^{-1}$ and that T has discrete spectrum $\{\lambda_k = \frac{(\mathrm{id}+\mu_k)}{\mu_k}\}$.

4. Conclusion

The paper has given a proof that the Laplacian resolvent operator is compact on the unit torus by means of the Fourier transform. We further found that given a compact Riemannian manifold, (M,g), that the embedding $H_0^1(M)\cap H_1^2(M)\to H_1^1(M)\to L^2(M)$ is compact.

For the Laplacian resolvent, $(\Delta+1)^{-1}$, we consequently showed that as a compact operator on the torus, it has discrete spectrum. Thus, $(\Delta+1)^{-1}$ satisfies the Rellich-Kondrakov theorem on T^n.

5. Conflict of Interest

There is no conflict of interest.

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